Surface Analysis of Realistic 3D Bile Canalicular Lumen

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Abstract
Curvature is an important property of surface. It can be used to estimate the surface energy and to characterize surfaces in three-dimensional space. In this project, for the first time, the local mean and local Gaussian curvatures for the 3D Bile canalicular geometries will be calculated in order to characterize the lumen of bile canaliculus.

Introduction
Curvature is an important property of surface. It can be used to distinguish between different types of surfaces by identifying local shapes such as: planar, concave, convex and saddle regions.

In 2D, curvature is defined as the rate of changing of the curve direction at a given point, Equation.1.

\[ \kappa = \left\| \frac{dT}{ds} \right\| \quad (1) \]

where \( \kappa \) is the curvature, \( T \) is tangential vector and \( ds \) is the infinitesimal arc length of the curve. It is equal to the inverse of the radius of the oscillating circle at a given point on the curve. In 3D, one can think of oscillating circles in planes that cross the surface at a point Fig.1. The maximum and minimum values of these curvatures are called principal curvatures. They are used to define the Gaussian and the mean curvatures as:

\[ G = (\kappa_1 \ast \kappa_2) \quad (2) \] Gaussian curvature

\[ H = \frac{1}{2}(\kappa_1 + \kappa_2) \quad (3) \] mean curvature

where \( \kappa_1 \) and \( \kappa_2 \) are the principal curvatures, \( G \) is the Gaussian curvature and \( H \) is the mean curvature.

The bending energy of the surface can be calculated using equation.4.

\[ G_{bend} = K_{b} \int ds H^{2} + \frac{K_{G}}{2} \int ds G. \quad (4) \]

Where \( K_{b} \) and \( K_{G} \) are bending rigidities associated with mean and Gaussian curvature, respectively. Gauss-Bonnet theorem relates the Gaussian curvature to the topology of the surface via the following equation:

\[ \int ds G = 4\pi (1 - g). \quad (5) \]

where "g" is the genus of the surface. Then, if the topology of the surface does not change, the only parameter that determines the energy of bending is mean curvature. Area Difference Elasticity model (ADE) shows a more complete relationship between the energy of bending and the mean curvature Equation.6:

\[ G_{bend} = \frac{K_{b}}{2} \left[ \int ds (H - C_{0})^{2} + \alpha (\Delta m - \Delta m_{0})^{2} \right] \quad (6) \]

where \( C_{0} \) is the spontaneous curvature and \( \Delta m \) is the total difference in the area between the outer and inner layers of a bilayer membrane. This equation was reported to...
successfully predict the different shapes of blood cells, i.e. stomatocyte, discocyte, echinocyte, [Lim H W G1 (Dec)].

Additionally, the local shapes of a surface can be determined by knowing mean and Gaussian curvatures. Table 1 shows the relationship between different shapes and the local curvatures.

<table>
<thead>
<tr>
<th>shape</th>
<th>Gaussian Curvature</th>
<th>mean curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>plane</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>cylinder</td>
<td>0</td>
<td>+/-</td>
</tr>
<tr>
<td>convex</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>concave</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>saddle</td>
<td>-</td>
<td>-/+</td>
</tr>
</tbody>
</table>

Surfaces are often represented by triangulated meshes. Estimating the curvature of a triangulated mesh is a challenging task in image processing and analysis. There are several methods of estimation. In general all of these methods can be categorized in to two group namely, surface-fitting methods, and discrete methods. The first group finds a function that locally fits the triangulated surface and then calculates the curvature of this function at the desired point [Gatzke and Grimm (2006)]. The second group uses discrete approximations for the analytical definition of curvature in continuous surfaces. The latter have the advantage of less computational costs and can be safely used for high resolution surfaces. Here, the Gaussian curvature will be calculated by angle deficit method. This method approximates the Gaussian curvature as $2\pi - \sum_{i=1}^{10} \alpha_i$ divided by an area associated with the vertex [Gatzke and Grimm (2006)] i.e.:

$$ G = \frac{(2\pi - \sum_{i=1}^{10} \alpha_i)}{S} \ (6) $$

and the mean curvature will be calculated using the following, [Nira Dyn and Levin (2000)]:

$$ H = \sum_{i=1}^{n} ||\vec{e}_i|| |\beta_i| \ (7) $$

where, $\vec{e}_i$ are the connecting edges and $\beta_i$ is the angle between the normals of the adjacent triangles, Fig.2.

### Methods

The meshes that were analyzed in this project were reconstructed from high resolution serial block face scanning electron microscopy data (SBF-SEM) provided by EM facility of MPI-CBG, [Ghaemi (2013)]. For SBF-SEM, the samples of the liver of a male mouse (C57bl/6jolahsd), were prepared by EM facility of MPI-CBG and Prof. Zerials lab. The images were acquired using an electron microscope Magellan 400 SEM from FEI that had a microtome 3ViewXP2-Gatan in its vacuum chamber.

### Results and Discussion

Fig 4 shows the distribution of raw data corresponding to the calculated mean curvatures in three bile canicular structures. After thresholding very large values (the values bigger than $5 \times 10^8 m^{-1}$ were removed) the histograms of mean curvature are shown in Fig.5. In both figures, one can easily see the similarities between these graphs. According to
Equation 6, this implies that the corresponding energies of bending must be also comparable between these surfaces.

Figure 4: **Distribution of mean curvature** The distribution of mean curvature (mm$^{-1}$) in three Bile canaliculi: S1, S2, S3. The red vertical lines are the median of mean curvature

![Figure 4: Distribution of mean curvature](image)

Table 2 shows the results of surface classification for the lumen of the same bile canalicular structures. From this table, it can be concluded that the lumen of a bile canaliculus is mostly composed of saddle surfaces, almost $50 \pm 3.0\%$. Convex and concave surfaces with a proportion of about $35 \pm 1.0\%$ come next. The remaining are plane and there is no cylinder ones.

<table>
<thead>
<tr>
<th></th>
<th>plane</th>
<th>convex</th>
<th>concave</th>
<th>saddle</th>
<th>cylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>0.1513</td>
<td>0.2088</td>
<td>0.1405</td>
<td>0.4995</td>
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<tr>
<td>S2</td>
<td>0.1756</td>
<td>0.1925</td>
<td>0.1472</td>
<td>0.4847</td>
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<tr>
<td>S5</td>
<td>0.0645</td>
<td>0.2198</td>
<td>0.1642</td>
<td>0.5515</td>
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</tr>
</tbody>
</table>

**Conclusion**

In this project, the mean and the Gaussian curvatures of the lumen of three bile canalicular structures were calculated. It was shown that the calculated mean curvatures were distributed in a similar pattern in all three cases. It implies that although these surfaces have different local geometrical features, the overall energy of bending is very similar for all of them. It was also shown that the lumen of all three bile canaliculi have similar geometrical features. It will be very useful to carry out the same type of surface analysis for a wide range of biliary geometries that are reconstructed from different parts of liver, or liver of different species, or species with different chemical treatments, in order to see the effects of these variants on the bending energy and the geometrical features of the lumen of bile canaliculi.

**References**


