Eigenvalue spectra of various operators using the PSE-DC formulation

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Abstract

Difficulties encountered with grid-based methods like finite differences, elements or volumes to simulate applications governed by a set of partial differential equations (PDEs) where mesh generation drastically increases the computational burden like those involving e.g. free surface flows in fluid dynamics, moving interfaces or complex geometries have motivated the development of meshfree methods like some type of particle methods. In their strong form to solve the set of PDEs, they require the accurate and efficient evaluation of spatial derivatives at various degrees. Within the frame of the particle strength exchange (PSE) formulation, a new promising formulation (PSE-DC) imposing moment conditions in a discrete form to obtain accuracy up to a certain order has been recently proposed [9].

Convergence rates and computational efficiencies of continuous and discretization corrected PSE operators as well as numerical dispersion and diffusion and stability conditions have been extensively studied in the latter reference. One additional approach to assess the relevance of the discretization method for dealing with given problems is to evaluate eigenspectra of operators. The evaluation and analysis of such spectra for the Laplacian, Grad-Div and Rot-Rot operators discretized here with this formulation are reported here. Optimally, only physically relevant and accurate eigensolutions are obtained [1]. Due to discretization, non-physical spurious eigenmodes may appear and can be coupled with the physical ones and deteriorate the accuracy of the computed solution in time-dependent problems.

Key words: Particle methods, Laplacian, Grad-Div, Rot-Rot operators, eigenvalue problem.
1 Introduction

Difficulties encountered with grid-based methods like finite differences, elements or volumes to simulate applications governed by a set of partial differential equations (PDEs) where mesh generation drastically increases the computational burden like those involving e.g. free surface flows in fluid dynamics, moving interfaces or complex geometries have motivated the development of meshfree methods like some type of particle methods. In their strong form to solve the set of PDEs, they require the accurate and efficient evaluation of spatial derivatives at various degrees. Within the frame of the particle strength exchange (PSE) formulation, Eldredge et al. proposed a uniform approach to determine these derivatives at any order [3], extending the method originally designed for treating the Laplacian operator in advection-diffusion problems [2] to a priori any kind of problem. Consistency of the discretized operators corresponding to the various derivatives imply that the interparticle spacing $h$ and the width of the operator kernel $\epsilon$ satisfy the constraint that their ratio $c = \frac{h}{\epsilon}$ tends to zero as $h$ and $\epsilon$ do. This condition can be removed if the moment conditions used to impose that all error terms vanish up to a certain order, are taken in a discrete form rather than in a continuous one [9]. It can then be shown that using discretization corrected (DC) kernels ensure that the overall error cannot be dominated by discretization errors.

The detailed formulation of PSE-DC operators and their link to finite differences are described in Ref. [9] and will only be briefly recalled hereafter. Convergence rates and computational efficiencies of continuous and discretization corrected PSE operators as well as numerical dispersion and diffusion and stability conditions have been extensively studied in the latter reference. In this report, we focus on the evaluation and analysis of the eigenspectrum of some operators used in various physical problems in discretized form as it may be important to determine the influence of the discretization method. Optimally, only physically relevant and accurate eigensolutions are obtained [1]. Due to discretization, non-physical spurious eigenmodes may appear and can be coupled with the physical ones and deteriorate the accuracy of the computed solution in time-dependent problems.

The report is organized as follows. The formulation that leads to the final eigenvalue problem that is to be solved to determine the spectra of various operators frequently encountered in physical problems, some test cases for the Laplacian, Grad-Div and Rot-Rot operators in one, two and three dimensions and the implementation are presented in section 2. The corresponding results are discussed in section 3. Finally conclusions are drawn and future prospects evoked in Section 4.
2 Problem setup

2.1 Equation discretization

In the PSE formulation, partial differential derivatives evaluated at location \( x \), such as:

\[
D^\beta u(x) = \frac{\partial |\beta| u(x)}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}
\]  

are approximated with discrete operators of the type;

\[
D_h^\beta u(x) = \frac{1}{\epsilon |\beta|} \sum_{p \in V(x)} v_p(u(x_p)) + (-1)^{|\beta|+1} u(x) \eta^\beta(x, x_p, \epsilon)
\]

In this equation, the symbols \( p \) and \( V(x) \) respectively denote the particle index and the set of particles that are in the neighborhood of \( x \) (\( |x_p - x| \leq r_c \), \( r_c \) cut-off radius). The quantity \( v_p = h^d \) is the volume of particle \( p \), \( d \) being the problem dimension. The function \( \eta^\beta(x, x_p, \epsilon) \) is given by:

\[
\eta^\beta(x, x_p, \epsilon) = \left( \sum_{|\gamma|=\alpha_{min}} a_\gamma(x) \left( \frac{x - x_p}{\epsilon} \right)^\gamma \right) e^{\frac{|x - x_p|}{\epsilon}}
\]

where \( r \) is the order of the approximation, \( \alpha_{min} \) is an integer equal to 1 if \( |\beta| \) is even and 0 otherwise and the coefficients \( a_\gamma(x) \) are determined by solving the following linear system of equations, resulting from the requirement that the error terms must vanish up to order \( r - 1 \):

\[
\sum_{|\gamma|=\alpha_{min}} a_\gamma(x) w_\alpha \gamma(x) = (-1)^{|\beta|} \beta! \delta_\alpha \beta
\]

for any vector \( \alpha \) such that \( \alpha_{min} \leq |\alpha| \leq |\beta| + r - 1 \). The quantity \( \delta_\alpha \beta \) is equal to 1 if the two vectors \( \alpha \) and \( \beta \) are identical, 0 otherwise. The weights \( w_\alpha \gamma(x) \) are expressed as:
\( w^\alpha \gamma(x) = \frac{1}{\epsilon |\alpha + \gamma| + d} \sum_{p \in V(x)} v_p(x - x_p)^{\alpha + \gamma} e^{\frac{|x - x_p|^2}{\epsilon}} \) \quad (5)

The linear system to be solved for the \( m \) coefficients \( a^\gamma(x) \) with \( m = C(|\beta| + r - 1 + d, d) - \alpha_{\text{min}} \) and \( C(p, q) \) is the number of combinations of \( p \) distinct things taken \( q \) at a time.

The eigenvalue problem of a given operator \( L \) to be solved for a prescribed domain \( V \) and associated boundary conditions is the following:

\[ Lu = \lambda u \] \quad (6)

where \( \lambda \) and \( u \) are respectively the corresponding eigenvalues and eigenfunctions of the operator for the associated problem. Applying a discretization method to this operator leads to the following discrete generalized eigenvalue problem:

\[ Au_h = \lambda_h B u_h \] \quad (7)

where \( A \) and \( B \) are matrices resulting from the discretization. The quantities \( \lambda_h \) and \( u_h \) are the discrete eigenvalues and eigenvectors corresponding to the discretization method. The discrete eigenspectrum is an approximation of the continuous one obtained by solving (6) by analytical means if this is possible. Matrix \( B \) can be the identity matrix if e.g. finite differences or PSE-DC methods are used or a mass matrix if a finite element method is used. For the PSE-DC method applied to operator \( D^\beta \), the coefficients \( A_{ij} \) are given by:

\[ A_{ij} = \frac{1}{\epsilon |\beta|} v_j \eta^\beta(x_i, x_j, \epsilon) \] \quad (8)

if \( i \neq j \) and \( j \in V(x_i) \).

\[ A_{ij} = 0 \] \quad (9)

if \( i \neq j \) and \( j \in V - V(x_i) \).

\[ A_{ii} = (-1)^{|\beta| + 1} \frac{\sum_{j \in V(x_i)} v_j \eta^\beta(x_i, x_j, \epsilon)}{\epsilon |\beta|} \] \quad (10)

4
These relationships directly come from the evaluation of the discretized formulation of \( D^3(u(x)) \) at particle locations \( x_i \). The extension to physical operators is straightforward as they are combinations of the derivatives \( D^3 \).

Boundary conditions are taken into account by modifying some lines of matrix \( A \). For instance, if Dirichlet conditions are considered, all the lines with indices \( i \) corresponding to the particles located on the given boundary must be filled with zeros \( (A_{ij} = 0) \) for all \( 1 \leq i \leq N_{\text{part}} \) where \( N_{\text{part}} \) is the number of particles. This generates zero eigenvalues when the system is solved.

### 2.2 Test cases

The eigenvalue problem associated with the laplacian operator (used e.g. in diffusion or fluid problems for computing the pressure) considered here is:

\[
\nabla \cdot (\nabla u) = \lambda u
\]

(11)
in the computational domain \( V = [0, 1]^d \) with homogeneous Dirichlet boundary conditions \( (u|_{\partial V} = 0) \) for \( d = 1, 2, 3 \). The quantity \( u \) can either be a scalar or a vector. This problem can be solved analytically and leads the following eigenvalues:

\[
\lambda_{\text{theo}} = -\left(k_x^2 + (1 - |\epsilon_{23d}|) k_y^2 + \delta_{3d} k_z^2\right) \pi^2
\]

(12)

where \( k_x, k_y \) and \( k_z \) are strictly positive integers and \( \epsilon_{ijk} \) and \( \delta_{ij} \) are the classical Levi-Civita (permutation) and Kronecker symbols. The corresponding modes vary as

\[
\sin(k_x \pi x) \sin(k_y \pi y)^{1-|\epsilon_{23d}|} \sin(k_z \pi z)^{\delta_{3d}}
\]

The Grad-Div and Rot-Rot operators appear in many physical problems like the ideal ocean wave one without friction and Coriolis forces, those involving the Maxwell equations, the Navier Stokes equations with a penalty formulation to account for incompressibility, ideal linear MHD equations, Navier equations within the theory of linear infinitesimal elasticity.

The eigenvalue problem for the Grad-Div operator is considered as:

\[
\nabla(\nabla \cdot u) = \lambda u
\]

(13)
in the computational domain \( V = [0, 1]^d \) with homogeneous boundary conditions (HBC):
\[ u \cdot n \mid_{\partial V} = 0 \quad (14) \]

for \( d = 2, 3 \), \( n \) being the outward normal vector. It leads to a spectrum composed by two parts: the first singular one with solenoidal modes \( u_S \) (for which \( \nabla \cdot u_S = 0 \)), all of them corresponding to the degenerate eigenvalue \( \lambda = 0 \) and the second one with irrotational modes \( u_I \) (for which \( \nabla \times u_I = 0 \)) associated with the non-zero distinct eigenvalues given by (12).

The existence of irrotational modes and expressions of the associated eigenvalues can be readily be shown by applying the Rot operator \( \nabla \times () \) to each term in equation (13) and using the identity \( \nabla \times (\nabla f) = 0 \) for all vectors \( f \). This clearly leads to the condition \( \lambda \nabla \times u = 0 \). Furthermore, the other identity \( \nabla \times (\nabla \times u) = \nabla (\nabla \cdot u) - \nabla \cdot (\nabla u) \) combined with the irrotationality condition \( \nabla \times u = 0 \) implies that the irrotational modes \( u_I \) satisfy the equation \( \nabla (\nabla \cdot u_I) = \nabla \cdot (\nabla u_I) \), which means that the corresponding eigenvalue equation for these modes can be reduced to (11). This problem has however different boundary conditions than the Laplacian one considered before, as imposing Dirichlet boundary conditions simultaneously for all components of \( u \) would lead to zero modes.

With the prescribed HBCs (14), the corresponding solenoidal and irrotational modes vary e.g. in 2D as \( u^T_S = (\sin(k\pi x) \cos(k\pi y), -\cos(k\pi x) \sin(k\pi y)) \) and \( u^T_I = (\sin(k_x \pi x) \cos(k_y \pi y), \cos(k_x \pi x) \sin(k_y \pi y)) \), where \( k_x, k_y \) and \( k \) are strictly positive integers. It can be easily shown that the eigenvalues associated with irrotationality are given by (12).

The eigenvalue problem for the Rot-Rot operator is considered as:

\[ \nabla \times (\nabla \times u) = \lambda u \quad (15) \]

in the computational domain \( V = [0, 1]^d \) with homogeneous boundary conditions (HBC):

\[ u \times n \mid_{\partial V} = 0 \quad (16) \]

for \( d = 2, 3 \), \( n \) being the outward normal vector. Here, the singular part of the spectrum is composed by irrotational modes \( u_I \), all of them corresponding to the degenerate eigenvalue \( \lambda = 0 \) whereas the solenoidal modes \( u_S \) are associated with the non-zero distinct eigenvalues given by the opposite of the ones obtained for the Laplacian operator (12). It is also possible to show easily that the solenoidal modes satisfy the equation \( \nabla \cdot (\nabla u_S) = -\lambda u_S \) and that
the full spectra (eigenvalues and eigenmodes) of the Grad-Div and Rot-Rot operators are identical in 2D.

2.3 Implementation

These test cases have been implemented in 1D with MATLAB (all problems reduce to the Laplacian one) and in 2D and 3D: by building a f90 client using the PPM library [6], [7]. The subroutine _dgegv_ of the LAPACK package [4] has been used to solve the corresponding discrete eigenvalue problem (7), with a QZ algorithm.

3 Results

As analytical expressions are available for the eigenvalues \( \lambda_{\text{theo}} \) of the continuous problem, relative errors \( \text{err} = \frac{|\lambda_{\text{theo}} - \lambda_{\text{comp}}|}{\lambda_{\text{theo}}} \) can be used to estimate the accuracy of the computed discrete spectra (\( \lambda_{\text{comp}} \)). Two parameters have a direct influence on the accuracy of this spectra: the approximation order \( r \) of the PSE-DC formulation and the particle average interspacing \( h \). The number of computed eigenvalues is clearly related to the latter. For a regular spacing (uniform distribution of particles), continuous eigenmodes with wave numbers larger than \( 1/h \) cannot be captured by the discretization. The accuracy of the eigenmodes clearly decreases as their order increases.

All the particle distributions in this study have been considered as uniform unless the opposite is explicitly specified in the text. The finite difference (FD) formulation can be recovered from the PSE-DC one by selecting a regular spacing \( h \) and a high value of the ratio \( c = \frac{h}{\epsilon} \), \( \epsilon \) being the kernel width. Analytical expressions for the kernel function \( \eta \) can be derived at various approximation order and the coefficients in matrix \( A \) in extensions of equation (1) have been compared for both FD and PSE-DC methods to test the implementation of the latter.

3.1 Laplacian operator

The accuracy of the first eigenvalues of the 1D spectra of the Laplacian operator for various values of \( h = \frac{1}{N_p - 1} \) and \( r \) can be seen in Fig. 1-4 for classical centered finite-difference stencils. As expected, the relative error decreases as the approximation order and particle number \( N_p \) increase. The r-th order
Fig. 1. Relative error on the first twenty computed eigenvalues w.r.t. theoretical eigenvalues, 1D laplacian operator, finite difference discretization with $N_p = 101$ points ($h = 0.01$); symbols: crosses ($r = 2$), circles ($r = 4$), squares ($r = 6$), diamonds ($r = 8$)

results obtained with the classical centered $(r + 1)$-point FD stencil can be reproduced exactly by taking $r_c = \frac{r}{2}$ (cf. Fig. 5 for the $r = 2$ case). Fig. 6 corresponds to a higher approximation ($r = 6$) and can readily be compared to Fig. 5. Accuracy is improved ad the approximation order is increased.

Another test consisted of analyzing the influence of the irregularity of the particle distribution. as shown in Fig. 7. The particle positions are taken as $x_i = (i - 1) \cdot (h + \Delta h \cdot R_i)$, where $R_i$ is a random number in $[-1, 1]$ and $\Delta h$ the maximal possible deviation from the regular grid. If $\Delta h$ vanishes, a uniform particle distribution on a regular grid is recovered. As the tolerated deviation is increased , $err$ increases. For $\Delta h \leq h$, the difference between the disturbed spectrum and the regular one remains quite small.

Fig. 8-11 enable one to observe the influence of $c$ on the relative error for various values of $r$ and $h$. For low values of $c$ (e.g. $c = 0.01$, accuracy is improved by increasing $1/h$ and $r$ whereas some saturation effect is visible on $err$ for large values of $c$ when $r$ increases. Plots are shown for $c = 1$ but the effect would be even stronger as $c$ gets larger. In that case, imposing higher approximation orders does not help in reducing the error and may even lead to large deviation from the theoretical spectrum if $c$ is large.
Fig. 2. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues vs. discretization order, 1D laplacian operator, finite difference discretization with $N_p = 101$ points ($h = 0.01$); symbols: crosses=1st ev, squares=2nd ev, circles=5th ev, triangles=10th ev, diamonds=20th ev.

This observation is valid for the 2D case as this can be seen in Fig. 12-15. For $c = 1$ and $r = 4$, the spectrum approximation is quite bad, which leads to the error visible in Fig. 13 whereas no deterioration is observed for $c = 0.01$ at the same order. For low values of $c$, i.e. $h < \epsilon$, increasing the number of particles improves the accuracy of the computed spectrum as shown in Fig. 16. for low values of $r$ ($r = 2$. The second-order results obtained with the classical centered five-point FD stencil can be reproduced exactly by taking $r_c = h$. for the various values of $h$. For other values of the cut-off radius $r_c$ ($r_c \leq \sqrt{2}h$), other FD stencils can be obtained with the particle formulation (e.g. a second-order nine-point stencil is obtained if $r_c = \sqrt{2}h$) and consistency can be checked. The corresponding accuracy (not shown here) is lower than the one obtained with the classical one but it remains of the same order. As it can be seen in Fig. 12, a satisfactory approximation of the spectrum can be obtained even with high values of $c$ if a second order error scheme ($r = 2$) is selected.

The influence of the cut-off parameter $r_c$ has also been investigated at approximations orders larger than 2. In all the simulated cases, increasing the cut-off radius leads to an increase of $err$ at a fixed approximation order. Furthermore, selecting a higher order of the approximation does not compensate for the loss of accuracy related to the increase of $r_c$ as shown in Fig. 17 for the whole
Fig. 3. Relative error on the first twenty computed eigenvalues w.r.t. theoretical eigenvalues, 1D laplacian operator, finite difference discretization with second-order stencil, symbols: crosses ($h = 0.05$), circles ($h = 0.02$), squares ($h = 0.01$), diamonds ($h = 0.005$).

spectrum and Fig. 18 for the first eigenvalues. Optimal accuracy seems to be obtained for intermediate values of $r$ and $r_c$.

The spectrum of the Laplacian operator on the given computational domain is computed correctly for $c = 0.01$ but not for $c = 1.0$ in several dimensions if the approximation order is larger than 2 as the system to be solved for the coefficients appearing in the kernel function $\eta$ becomes very stiff.

3.2 Grad-Div operator

As noticed in section 2.2, this operator contains a physical constraint that induces the presence of degenerate eigenmodes. As opposed to the discrete irrotational modes, the solenoidal ones may generate numerical difficulties when the operator is discretized with various general methods unless some specific approach is used[1]. In the context of finite elements, the COOL hp method enables on to obtain . Otherwise, the infinitely degenerate divergence-free eigensolutions expand to a discrete spectrum that changes with the approximation order and mesh refinement. This is what is observed also with the PSE-DC method if the same grid is used for both components of $\mathbf{u}$ as shown in
Fig. 4. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues vs. number of grid points, 1D laplacian operator, finite difference discretization with second-order centered three-point stencil; symbols: crosses=1st ev, squares=2nd ev, circles=5th ev, triangles=10th ev.

Fig. 19 and 20, where spectra are shown for various approximation orders and numbers of regularly distributed particles (i.e. spacing $h$) for a given value of the spacing to kernel width ratio $c$. Similar plots are obtained for higher values of $c$. If $c = 1$, the matrix $A$ could not be built above $r = 4$ and therefore the spectra could not be computed. As opposed to the Laplacian case, some of the computed eigenvalues have non-zero imaginary parts. As it can be observed in Fig. 21, their number decreases as $c$ increases.

When dealing with Maxwell’s equation, it is very common to use staggered grids to impose suitable boundary conditions e.g. [8],[10]. This implies that the components of the electric or magnetic fields are computed at different points in a computational cell but does not cure the emergence of the spurious eigenmodes. Two sets of particles corresponding to each component of the eigenvector $u$ have been used to generate the coefficients of matrix $A$. The coefficients $A_{ij}$ corresponding to straight derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are computed by formulae (8)-(10) using the set of particles to which particle $i$ belongs, while the cross derivative $\frac{\partial}{\partial x\partial y}$ is evaluated by considering particles $j$ in the other set to mimic classical finite-difference schemes with staggered grids.

It is of course possible to get rid of the divergence free modes by solving equation (11) for irrotational modes (i.e. by adding the constraint $\nabla \times u = 0$)
Fig. 5. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues, 1D laplacian operator, PSE-DC discretization with uniform distribution of particles, r=2, c=1, \( r_c = h \); symbols: crosses \((h = 0.05)\), circles \((h = 0.02)\), squares \((h = 0.01)\), diamonds \((h = 0.005)\).

with the associated homogeneous boundary conditions (HBC) instead of (13) for all the modes.

3.3 Rot-Rot operator

As noticed in section 2.2, the spectra for the Grad-Div and Rot-Rot operators with the given domain and respective boundary conditions are identical in 2D. This has been checked numerically.

4 Conclusion

A preliminary study of the computed eigenspectra of the Laplacian, Grad-Div and Rot-Rot operators in 1D and 2D for a square geometry and selected Dirichlet conditions, using the PSE-DC method, is reported. Analytical expressions for the eigenvalues are available for comparison with computed values. Al-
Fig. 6. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues, 1D laplacian operator, PSE-DC discretization with uniform distribution of particles, \( r=6, c=1, r_c = h \); symbols: crosses \((h = 0.05)\), circles \((h = 0.02)\), squares \((h = 0.01)\), diamonds \((h = 0.005)\).

though the expected exponential convergence is observed for low values of the particle spacing to kernel width ratio in the Laplacian case, some limitations in the use of large values of \( c \) have been observed as the system to be solved for the terms appearing in the kernel function \( \eta \) becomes very stiff. In the Grad-Div (resp. Rot-Rot) case in 2D, the computed spectra are strongly polluted by the degenerate modes corresponding to the divergence-free (resp. rotational-free) condition. The situation does not improve if the particle density or the approximation order is increased for any value of \( c \). The same limitation in the use of large values of \( c \) applies.

References


Fig. 7. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues, 1D laplacian operator, PSE-DC discretization with random distribution of \( N_p = 201 \) particles, \( r=2, c=1, \Delta c = 4 \times h \); symbols: crosses (\( \Delta h = 0 \) ), diamonds (\( \Delta h = 0.5 \times h \)), circles (\( \Delta h = 2 \times h \)), squares (\( \Delta h = 5 \times h \)).


Fig. 8. Relative error on the first twenty computed eigenvalues w.r.t. theoretical eigenvalues, 1D laplacian operator, PSE-DC discretization, $c=1$, $N_p = 201$ particles uniformly distributed with spacing $h$, $r_c = 2 \ast r \ast h$; symbols: crosses ($r=2$), circles ($r=4$), squares ($r=6$) diamonds ($r=8$).
Fig. 9. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues vs. error order $N$, 1D laplacian operator, PSE-DC discretization, $c=1$, $N_p = 201$ particles uniformly distributed with spacing $h$, $r_c = 2 \cdot N \cdot h$; symbols: crosses=1st ev, squares=2nd ev, circles=5th ev, triangles=10th ev, diamonds=20th ev.
Fig. 10. Relative error on the first twenty computed eigenvalues w.r.t. theoretical eigenvalues, 1D laplacian operator, PSE-DC discretization, c=0.01, $N_p = 201$ particles uniformly distributed with spacing $h$, $r_c = 2 * r * h$; symbols: crosses ($r=2$), circles ($r=4$), squares ($r=6$), diamonds ($r=8$).
Fig. 11. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues vs. error order $N$, 1D laplacian operator, PSE-DC discretization, $c=0.01$, $N_p = 201$ particles uniformly distributed with spacing $h$, $r_c = 2 \times r \times h$; symbols: crosses=1st ev, squares=2nd ev, circles=5th ev, triangles=10th ev, diamonds=20th ev.
Fig. 12. Relative error on the first 81 computed eigenvalues (ev) w.r.t theoretical eigenvalues, 2D laplacian operator, PSE-DC discretization, r=2, c=1.0, particles uniformly distributed with spacing h, \( r_c = h \); symbols: squares \((N_p = 11 \times 11)\), crosses \((N_p = 21 \times 21)\), circles \((N_p = 51 \times 51)\).
Fig. 13. Relative error on the first 53 computed eigenvalues (ev) w.r.t theoretical eigenvalues, 2D laplacian operator, PSE-DC discretization, $r=4$, $c=1.0$, particles uniformly distributed with spacing $h$, $r_c = 4 \times h$; symbols: squares ($N_p = 11 \times 11$), crosses ($N_p = 21 \times 21$) circles ($N_p = 51 \times 51$).
Fig. 14. Relative error on the first 81 computed eigenvalues (ev) w.r.t theoretical eigenvalues, 2D laplacian operator, PSE-DC discretization, r=2, c=0.01, particles uniformly distributed with spacing h, $r_c = \sqrt{2} \times h$; symbols: squares ($N_p = 11 \times 11$), crosses ($N_p = 21 \times 21$) circles ($N_p = 51 \times 51$).
Fig. 15. Relative error on the first 81 computed eigenvalues (ev) w.r.t theoretical eigenvalues, 2D laplacian operator, PSE-DC discretization, r=4, c=0.01, particles uniformly distributed with spacing h, $r_c = 4 \times h$; symbols: squares ($N_p = 11 \times 11$), crosses ($N_p = 21 \times 21$) circles ($N_p = 51 \times 51$).
Fig. 16. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues vs. number of particles in each direction, 2D laplacian operator, PSE-DC discretization, c=0.01, particles uniformly distributed with spacing h, $r_c = \sqrt{2} \times h$; symbols: squares=1st ev, crosses=2nd ev, circles=3rd ev, diamonds=4th ev, triangles=5th ev.
Fig. 17. Relative error on the first 81 computed eigenvalues (ev) w.r.t theoretical eigenvalues, 2D laplacian operator, PSE-DC discretization, $N_p = 51 \times 51$, $c=0.01$, particles uniformly distributed with spacing $h$; symbols: squares ($r_c = \sqrt{2} \times h$, $r=2$), crosses ($r_c = 4 \times h$, $r=4$) circles ($r_c = 6 \times h$, $r=6$).
Fig. 18. Relative error on computed eigenvalues (ev) w.r.t. theoretical eigenvalues vs. approximation order, 2D laplacian operator, PSE-DC discretization, $c=0.01$, $N_p = 51 \times 51$ particles uniformly distributed with spacing $h$ in each direction; symbols: squares=1st ev, crosses=2nd ev, circles=3rd ev, diamonds=4th ev, triangles=5th ev.
Fig. 19. Modules of the eigenvalues of the Grad-Div operator on $V = [0, 1]^2$ with homogeneous boundary conditions $u \cdot n \mid_{\partial V} = 0$ for various approximation orders, PSE-DC discretization, $c=0.01$, $N_p = 11 \times 11$ particles uniformly distributed with spacing $h$ in each direction, $r_c = 2h$; symbols: stars=theoretical irrotational spectrum, crosses=computed spectrum.
Fig. 20. Modules of the eigenvalues of the Grad-Div operator on $V = [0, 1]^2$ with homogeneous boundary conditions $\mathbf{u} \cdot \mathbf{n} \mid_{\partial V} = 0$ for various uniform particle distributions ($N$ particles in each direction), PSE-DC discretization, $c=0.01$, $r = 2$, $r_c = 2h$; symbols: stars=theoretical irrotational spectrum, crosses=computed spectrum.
Fig. 21. Eigenvalue spectra of the Grad-Div operator on $V = [0, 1]^2$ with homogeneous boundary conditions $u \cdot n |_{\partial V} = 0$ for various values of $c$; uniform particle distributions ($N = 21$ particles in each direction), PSE-DC discretization, $r = 2$; symbols: stars=theoretical irrotational spectrum, crosses=computed spectrum with $c = 0.01$, $r_c = 2$, circles= computed spectrum with $c = 1$, $r_c = 3h$. 