Question 1: Calculations with operators
To train your skills in calculating with the operators in vector analysis you have been introduced to in the lecture, please prove the following statements. Let \( \mathbf{v} \) be a vector field and \( f \) a scalar field

a) \[ \text{div}(f \mathbf{v}) = \mathbf{v} \cdot \text{grad} f + f \text{ div } \mathbf{v} \]  

b) \[ \text{div} \, \text{curl} \, \mathbf{v} = 0 \]  

c) \[ \text{curl} \, \text{curl} \, \mathbf{v} = \text{grad} \, \text{div} \, \mathbf{v} - \Delta \mathbf{v} \]  

d) \[ \text{div} \left( \mathbf{v}_1 \times \mathbf{v}_2 \right) = \mathbf{v}_2 \cdot \text{curl} \, \mathbf{v}_1 - \mathbf{v}_1 \cdot \text{curl} \, \mathbf{v}_2 \]

Question 2: Rotation of a rigid body
Consider a rotating rigid body with rotation axis in the origin \( O \). Let the position vector be \( \mathbf{r} = (x, y, z) \) and the angular velocity \( \omega = (\omega_1, \omega_2, \omega_3) \).

a) What is the velocity field \( \mathbf{v} \) of the rigid body?

b) Compute \( \text{curl} \, \mathbf{v} \). Given your result can you tell what quantity the operator \( \text{curl} \) is actually measuring?

Question 3: Flux in a Coulomb field
Consider an electric point charge \( e \) in the origin \( O \) of a cartesian coordinate system. Let \( \mathbf{v}(\mathbf{r}) \) be the corresponding electric Coulomb field with

\[ \mathbf{v}(\mathbf{r}) = C \frac{e}{r^3} \mathbf{r} \]

with \( \mathbf{r} = (x, y, z)^T \) and \( r = \sqrt{x^2 + y^2 + z^2} \).
Calculate the flux \( \phi \) of a point charge through a sphere with radius \( R \) and origin \( O \).
Question 4: Potential fields

Let \( \mathbf{v} \) be a potential field with potential \( f \).

a) Show that \( \mathbf{v} \) is vortex-free.

*Hint:* Recall the definition of a potential field and your calculations in the self-test questions.

b) Let \( \mathbf{v}(\mathbf{r}) \) be a Coulomb field with

\[
\mathbf{v}(\mathbf{r}) = -\frac{C \mathbf{r}}{r^3}
\]

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with \( \mathbf{r} = (x, y, z)^T \) and \( r = \sqrt{x^2 + y^2 + z^2} \) in cartesian coordinates. Show that \( \text{curl} \mathbf{v} = \mathbf{0} \).