Question 1: Calculations with operators

Let \( \mathbf{v}(x, y, z) = \begin{pmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{pmatrix} \) be a vector field and \( f = f(x, y, z) \) a scalar field.

The question implicitly assumed sufficient smoothness of \( f \) and \( \mathbf{v} \) as stated in the answers to the subquestions.

a) Show that: \( \text{div}(f \mathbf{v}) = \mathbf{v} \cdot \text{grad} f + f \text{div} \mathbf{v} \).

It is necessary that \( f \) and \( \mathbf{v} \) are both (at least once) differentiable.

Application of the product rule for differentiation of scalar functions to each of the components yields

\[
\text{div}(f \mathbf{v}) = \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)
\]
\[
= f \frac{\partial}{\partial x}(v_1) + v_1 \frac{\partial}{\partial x}(f) + f \frac{\partial}{\partial y}(v_2) + v_2 \frac{\partial}{\partial y}(f) + f \frac{\partial}{\partial z}(v_3) + v_3 \frac{\partial}{\partial z}(f)
\]
\[
= v_1 \frac{\partial}{\partial x}(f) + v_2 \frac{\partial}{\partial y}(f) + v_3 \frac{\partial}{\partial z}(f) + f \frac{\partial}{\partial x}(v_1) + f \frac{\partial}{\partial y}(v_2) + f \frac{\partial}{\partial z}(v_3)
\]
\[
= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \left( \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \right) + f \left( \frac{\partial}{\partial x}(v_1) + \frac{\partial}{\partial y}(v_2) + \frac{\partial}{\partial z}(v_3) \right)
\]
\[
= \mathbf{v} \cdot \text{grad} f + f \text{div} \mathbf{v}
\]

Note that the statement holds true for any differentiable \( n \)-dimensional vector field \( \mathbf{v} \), the proof then simply includes \( n \) summands, one for each coordinate direction.

b) Show that: \( \text{div} \text{curl} \mathbf{v} = 0 \). \( \text{“The curl is source-free.”} \)

It is necessary that \( \mathbf{v} \) is twice differentiable and second order derivatives can be interchanged in order. The latter is satisfied p.e. when \( \mathbf{v} \) is twice \( \textit{continuously} \) differentiable (Schwarz’s theorem).
\[
div \, \curl \, \mathbf{v} = \dfrac{\partial}{\partial \mathbf{x}} \left( \dfrac{\partial}{\partial y} v_3 - \dfrac{\partial}{\partial z} v_2 \right) - \dfrac{\partial}{\partial \mathbf{y}} \left( \dfrac{\partial}{\partial z} v_1 - \dfrac{\partial}{\partial x} v_3 \right) + \dfrac{\partial}{\partial \mathbf{z}} \left( \dfrac{\partial}{\partial x} v_2 - \dfrac{\partial}{\partial y} v_1 \right) \]
\[
= \dfrac{\partial^2}{\partial x^2} v_3 - \dfrac{\partial^2}{\partial y^2} v_2 + \dfrac{\partial^2}{\partial z^2} v_1 - \dfrac{\partial^2}{\partial y^2} v_3 + \dfrac{\partial^2}{\partial z^2} v_2 - \dfrac{\partial^2}{\partial x^2} v_1 \]
\[
= 0
\]

\(\text{c) Show that: } \curl \, \curl \, \mathbf{v} = \grad \, \div \, \mathbf{v} - \Delta \mathbf{v}\)

Again, it is necessary that \(\mathbf{v}\) is twice differentiable and second order derivatives can be interchanged in order. The latter is satisfied p.e. when \(\mathbf{v}\) is twice continuously differentiable (Schwarz’s theorem).

First, expand both \(\curl\) operators into the coordinate form, then add a clever zero like \(+ \dfrac{\partial^2}{\partial z^2} v_1 - \dfrac{\partial^2}{\partial x^2} v_1\) in the \(x\)-coordinate to get \(\Delta v_1\).

\[
\curl \, \curl \, \mathbf{v} = \curl \left( \begin{array}{c}
\dfrac{\partial}{\partial y} v_3 - \dfrac{\partial}{\partial z} v_2 \\
\dfrac{\partial}{\partial z} v_1 - \dfrac{\partial}{\partial x} v_3 \\
\dfrac{\partial}{\partial x} v_2 - \dfrac{\partial}{\partial y} v_1
\end{array} \right)
\]
\[
= \left( \begin{array}{c}
\dfrac{\partial}{\partial \mathbf{x}} v_3 - \dfrac{\partial}{\partial \mathbf{y}} v_2 \\
\dfrac{\partial}{\partial \mathbf{y}} v_1 - \dfrac{\partial}{\partial \mathbf{z}} v_3 \\
\dfrac{\partial}{\partial \mathbf{z}} v_2 - \dfrac{\partial}{\partial \mathbf{x}} v_1
\end{array} \right) \times \left( \begin{array}{c}
\dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 - \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_2 \\
\dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_1 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_3 \\
\dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2 - \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_1
\end{array} \right)
\]
\[
= \left( \begin{array}{c}
\dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_2 + \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_1 - \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{z}} v_1 - \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_1 + \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_1 \\
\dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_2 + \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_2 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_1 + \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_1 \\
\dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_1 - \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 - \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 + \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2
\end{array} \right)
\]
\[
= \left( \begin{array}{c}
\dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_2 + \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_1 + \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{z}} v_1 - \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_1 + \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_1 \\
\dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_2 + \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_2 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_1 + \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{y}} v_1 \\
\dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_1 - \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 + \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 - \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2
\end{array} \right)
\]
\[
= \left( \begin{array}{c}
\dfrac{\partial}{\partial \mathbf{x}} v_2 + \dfrac{\partial}{\partial \mathbf{y}} v_3 + \dfrac{\partial}{\partial \mathbf{z}} v_1 - \Delta v_1 \\
\dfrac{\partial}{\partial \mathbf{y}} v_3 + \dfrac{\partial}{\partial \mathbf{z}} v_1 + \dfrac{\partial}{\partial \mathbf{x}} v_2 - \Delta v_2 \\
\dfrac{\partial}{\partial \mathbf{z}} v_1 + \dfrac{\partial}{\partial \mathbf{y}} v_2 + \dfrac{\partial}{\partial \mathbf{x}} v_3 - \Delta v_3
\end{array} \right)
\]
\[
= \left( \begin{array}{c}
\dfrac{\partial}{\partial \mathbf{x}} v_2 + \dfrac{\partial}{\partial \mathbf{y}} v_3 + \dfrac{\partial}{\partial \mathbf{z}} v_1 \\
\dfrac{\partial}{\partial \mathbf{y}} v_3 + \dfrac{\partial}{\partial \mathbf{z}} v_1 \\
\dfrac{\partial}{\partial \mathbf{z}} v_1
\end{array} \right) \frac{\partial^2}{\partial \mathbf{x} \partial \mathbf{y}} v_1 + \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_3 - \dfrac{\partial^2}{\partial \mathbf{z} \partial \mathbf{x}} v_2 + \dfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{z}} v_1 - \dfrac{\partial^2}{\partial \mathbf{y} \partial \mathbf{z}} v_1
\]
\[
= \grad \, \div \, \mathbf{v} - \Delta \mathbf{v}
\]

\(\text{d) Show that: } \div (\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{v}_2 \cdot \curl \mathbf{v}_1 - \mathbf{v}_1 \cdot \curl \mathbf{v}_2.\)
To define the curl’s, $\mathbf{v}_1, \mathbf{v}_2$ need to be (at least once) differentiable.

Let $\mathbf{v}_1(x, y, z) = \begin{pmatrix} v_{11}(x, y, z) \\ v_{12}(x, y, z) \\ v_{13}(x, y, z) \end{pmatrix}$ and $\mathbf{v}_2(x, y, z) = \begin{pmatrix} v_{21}(x, y, z) \\ v_{22}(x, y, z) \\ v_{23}(x, y, z) \end{pmatrix}$.

\[
div(\mathbf{v}_1 \times \mathbf{v}_2) = div \begin{pmatrix} v_{12}v_{23} - v_{13}v_{22} \\ v_{13}v_{21} - v_{11}v_{23} \\ v_{11}v_{22} - v_{12}v_{21} \end{pmatrix} = \frac{\partial}{\partial x} (v_{12}v_{23} - v_{13}v_{22}) + \frac{\partial}{\partial y} (v_{13}v_{21} - v_{11}v_{23}) + \frac{\partial}{\partial z} (v_{11}v_{22} - v_{12}v_{21}) \\
= v_{12} \frac{\partial}{\partial x} v_{23} + v_{23} \frac{\partial}{\partial x} v_{12} - v_{13} \frac{\partial}{\partial x} v_{22} - v_{22} \frac{\partial}{\partial x} v_{13} \\
+ v_{21} \frac{\partial}{\partial y} v_{13} + v_{13} \frac{\partial}{\partial y} v_{21} - v_{11} \frac{\partial}{\partial y} v_{23} - v_{23} \frac{\partial}{\partial y} v_{11} \\
+ v_{22} \frac{\partial}{\partial z} v_{11} + v_{11} \frac{\partial}{\partial z} v_{22} - v_{12} \frac{\partial}{\partial z} v_{21} - v_{21} \frac{\partial}{\partial z} v_{12} \\
= (v_{21} \left( \frac{\partial}{\partial y} v_{13} - \frac{\partial}{\partial z} v_{12} \right) + v_{22} \left( \frac{\partial}{\partial z} v_{11} - \frac{\partial}{\partial x} v_{13} \right) + v_{23} \left( \frac{\partial}{\partial x} v_{12} - \frac{\partial}{\partial y} v_{11} \right)) \\
- (v_{11} \left( \frac{\partial}{\partial y} v_{23} - \frac{\partial}{\partial z} v_{22} \right) + v_{12} \left( \frac{\partial}{\partial z} v_{21} - \frac{\partial}{\partial x} v_{23} \right) + v_{13} \left( \frac{\partial}{\partial x} v_{22} - \frac{\partial}{\partial y} v_{21} \right)) \\
= v_2 \cdot \text{curl} \mathbf{v}_1 - \mathbf{v}_1 \cdot \text{curl} \mathbf{v}_2
\]

Aside: The Levi-Civita symbol

Cross products may also be written with the Levi-Civita symbol or Epsilon tensor $\epsilon_{ijk}$. It is defined as

\[
\epsilon_{ijk} = \begin{cases} 
1 & \text{if } ijk \text{ is an even permutation, e.g. 123} \\
-1 & \text{if } ijk \text{ is an odd permutation, e.g. 321} \\
0 & \text{if any indices repeat}
\end{cases}
\]

For example,

\[
\epsilon_{231} = -\epsilon_{132} = -(-\epsilon_{123}) = 1,
\]

so by switching indices, one can quickly arrive at a known value for $\epsilon$. It can be generalized to $n$ dimensions as

\[
\epsilon_{a_1a_2...a_n} = \prod_{1 \leq i < j \leq n} \text{sgn}(a_j - a_i),
\]

although the index switching described above might be faster for calculations.

With $\epsilon_{ijk}$, a cross product of two vectors $\vec{a}, \vec{b}$ can be written as

\[
(\vec{a} \times \vec{b})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a^j b^k,
\]
where $\vec{e}_i$ is the i'th orthogonal basis vector. Equipped with this knowledge, some identities involving the $\nabla$ operator are quite quick to prove:

Let’s take $\nabla \cdot (\nabla \times \vec{v})$ as an example:

$$\nabla \cdot (\nabla \times \vec{v}) = \partial_i \vec{e}_i (\epsilon_{ijk} \partial_j v_k \vec{e}_k)$$  \hspace{1cm} (4)

$$= \partial_i \epsilon_{ijk} \partial_j v_k \vec{e}_i \vec{e}_k$$  \hspace{1cm} (5)

$$= 0,$$  \hspace{1cm} (6)

as $\vec{e}_i \vec{e}_k = \delta_{ik}$, meaning that their scalar product equals 0 (they’re orthogonal!) if they are not the same two vectors. Now $\epsilon_{ijk}$ vanishes when two indices coincide while $\delta_{ik}$ is only non-zero if they do. Therefore, their product has to be zero. This is by the way a very general result: The product of any anti-symmetric tensor with a symmetric one is bound to be zero. In this example we have also used the Einstein summation convention, where a sum over repeated indices is implied.
**Question 2: Rotation of a rigid body**

Let us consider a rotating rigid body with rotation axis in the origin $O$. Let the position vector be $\mathbf{r} = (x, y, z)$ and the angular velocity $\mathbf{\omega} = (\omega_1, \omega_2, \omega_3)$.

a) Angular velocity of $\mathbf{\omega} = (\omega_1, \omega_2, \omega_3)$ implies by the right-hand rule that the velocity field $\mathbf{v}$ of the rigid body is:

$$\mathbf{v} = \mathbf{\omega} \times \mathbf{r} = \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix}$$

b)

$$\text{curl} \mathbf{v} = \text{curl} \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y} (\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z} (\omega_3 x - \omega_1 z) \\ \frac{\partial}{\partial z} (\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x} (\omega_1 y - \omega_2 x) \\ \frac{\partial}{\partial x} (\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y} (\omega_2 z - \omega_3 y) \end{pmatrix}$$

$$= \begin{pmatrix} \omega_1 + \omega_1 \\ \omega_2 + \omega_2 \\ \omega_3 + \omega_3 \end{pmatrix}$$

So $\text{curl}$ actually measures the angular velocity. In the case of the rigid body, it is twice the angular velocity.

Moreover for the rotation of a rigid body, the $\text{curl}$ of the velocity field which is itself a vector field, turns out to be constant in space.

**Question 3: Flux in a Coulomb field**

Consider a electric point charge $e$ in the origin $O$ of a cartesian coordinate system. Let $\mathbf{v}(\mathbf{r})$ be the corresponding electric Coulomb field with

$$\mathbf{v}(\mathbf{r}) = C \frac{e}{r^3} \mathbf{r}$$

with $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Then, the flux $\phi$ of a point charge through a sphere with radius $R$ and surface $S$ is given by:

$$\phi = \int_S C \frac{e}{R^3} \mathbf{r} \cdot \mathbf{n} dS$$

With $\mathbf{r}$ and $\mathbf{n}$ parallel we get $\mathbf{r} \cdot \mathbf{n} = R$ on the sphere. Hence,

$$\phi = C \frac{e}{R^3} \int_S dS = C \frac{e}{R^2} 4\pi R^2 = 4\pi Ce$$

So the flux is proportional to the charge $e$ and independent of the radius.
Note that we may not use Gauss’ theorem, because \( \mathbf{v}(\mathbf{r}) \) is not continuously differentiable in the origin. In fact, \( \mathbf{v}(\mathbf{r}) \) is not continuous in the origin, hence it cannot be differentiable. However, off the origin we have

\[
\text{div} \, \mathbf{v}(\mathbf{r}) = Ce \left[ \frac{\partial}{\partial x} \frac{x}{r^3} + \frac{\partial}{\partial y} \frac{y}{r^3} + \frac{\partial}{\partial z} \frac{z}{r^3} \right] \\
= Ce \left[ \frac{r^3 - x \frac{3}{2} r \cdot 2x}{r^6} + \frac{r^3 - y \frac{3}{2} r \cdot 2y}{r^6} + \frac{r^3 - z \frac{3}{2} r \cdot 2z}{r^6} \right] \\
= Ce \left[ \frac{1}{r^6} (3r^3 - 3(x^2 + y^2 + z^2) r) \right] \\
= 0
\]

So for any region \( B \) not containing the origin with sufficiently smooth boundary \( \partial B \) we conclude by Gauss’ theorem

\[
\int_{\partial B} \frac{Ce}{R^3} \mathbf{r} \cdot \mathbf{n} \, dS = \int_{\partial B} \mathbf{v}(\mathbf{r}) \cdot \mathbf{n} \, dS = \int_B \text{div} \, \mathbf{v}(\mathbf{r}) \, dV = \int_B 0 \, dV = 0.
\]

For example, this applies to the region obtained by removing an inner ball of radius \( R_i \) from a ball of outer radius \( R_o \). The flux through this region’s boundary is zero by Gauss’ theorem, and also zero because the influx at the inner surface (\( 4\pi Ce \) by the direct calculation) is exactly balanced by the outflux at the outer surface (\( 4\pi Ce \) as well).
Question 4: Potential fields

Let \( \mathbf{v} \) be a potential field with potential \( f \). This implies that

\[
\mathbf{v} = \text{grad} \, f
\]

a) In the self-test question, you showed that \( \text{curl} \, \text{grad} \, f = 0 \). Hence, it is easy to show that \( \mathbf{v} \) is vortex-free, namely:

\[
\text{curl} \, \mathbf{v} = \text{curl} \, \text{grad} \, f = 0
\]

This is stated in words as “Potential fields are vortex-free”.

b) Let \( \mathbf{v}(r) \) be a Coulomb field with

\[
\mathbf{v}(r) = -C \frac{\mathbf{r}}{r^3}
\]

with \( \mathbf{r} = (x, y, z)^T \) and \( r = \sqrt{x^2 + y^2 + z^2} \) in cartesian coordinates. So \( \mathbf{v} \) is also given by:

\[
\mathbf{v}(r) = -C \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)^T
\]

Then

\[
\text{curl} \, \mathbf{v}(r) = -C \text{curl} \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)
\]

\[
= -C \begin{vmatrix}
\frac{\partial}{\partial y} \frac{z}{r^3} & -\frac{\partial}{\partial z} \frac{y}{r^3} \\
\frac{\partial}{\partial z} \frac{x}{r^3} & -\frac{\partial}{\partial x} \frac{z}{r^3} \\
\frac{\partial}{\partial x} \frac{y}{r^3} & -\frac{\partial}{\partial y} \frac{x}{r^3}
\end{vmatrix}
\]

\[
= -C \begin{pmatrix}
-\frac{z \partial}{\partial y} + \frac{y \partial}{\partial z} \\
-\frac{x \partial}{\partial z} + \frac{z \partial}{\partial x} \\
-\frac{y \partial}{\partial x} + \frac{x \partial}{\partial y}
\end{pmatrix}
\]

\[
= 0
\]