Question 1: Calculations with operators

Let \( \mathbf{v}(x, y, z) = \begin{pmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{pmatrix} \) be a vector field and \( f(x, y, z) \) a scalar field.

The question implicitly assumed sufficient smoothness of \( f \) and \( \mathbf{v} \) as stated in the answers to the subquestions.

a) Show that: \( \text{div}(f \mathbf{v}) = \mathbf{v} \cdot \text{grad} f + f \text{div} \mathbf{v} \).

It is necessary that \( f \) and \( \mathbf{v} \) are both (at least once) differentiable.

Application of the product rule for differentiation of scalar functions to each of the components yields

\[
\text{div}(f \mathbf{v}) = \frac{\partial}{\partial x} (fv_1) + \frac{\partial}{\partial y} (fv_2) + \frac{\partial}{\partial z} (fv_3)
\]

\[
= f \frac{\partial}{\partial x} (v_1) + v_1 \frac{\partial}{\partial x} (f) + f \frac{\partial}{\partial y} (v_2) + v_2 \frac{\partial}{\partial y} (f) + f \frac{\partial}{\partial z} (v_3) + v_3 \frac{\partial}{\partial z} (f)
\]

\[
= v_1 \frac{\partial}{\partial x} (f) + v_2 \frac{\partial}{\partial y} (f) + v_3 \frac{\partial}{\partial z} (f) + f \frac{\partial}{\partial x} (v_1) + f \frac{\partial}{\partial y} (v_2) + f \frac{\partial}{\partial z} (v_3)
\]

\[
= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} + f \left( \frac{\partial}{\partial x} (v_1) + \frac{\partial}{\partial y} (v_2) + \frac{\partial}{\partial z} (v_3) \right)
\]

\[
= \mathbf{v} \cdot \text{grad} f + f \text{div} \mathbf{v}
\]

Note that the statement holds true for any differentiable \( n \)-dimensional vector field \( \mathbf{v} \), the proof then simply includes \( n \) summands, one for each coordinate direction.
b) Show that: $\text{div \ curl} \mathbf{v} = 0$. (“The curl is source-free.”)

It is necessary that $\mathbf{v}$ is twice differentiable and second order derivatives can be interchanged in order. The latter is satisfied p.e. when $\mathbf{v}$ is twice \textit{continuously} differentiable (Schwarz’s theorem).

$$
\text{div \ curl} \mathbf{v} =
\begin{vmatrix}
\frac{\partial}{\partial y} v_3 - \frac{\partial}{\partial z} v_2 \\
\frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3 \\
\frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1 \\
\end{vmatrix}
$$

$$
= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} v_3 - \frac{\partial}{\partial z} v_2 \right) + \frac{\partial}{\partial y} \left( \frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3 \right) + \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1 \right)
$$

$$
= \frac{\partial^2}{\partial x \partial y} v_3 - \frac{\partial^2}{\partial x \partial z} v_2 + \frac{\partial^2}{\partial y \partial z} v_1 - \frac{\partial^2}{\partial y \partial x} v_3 + \frac{\partial^2}{\partial z \partial x} v_2 - \frac{\partial^2}{\partial z \partial y} v_1
$$

$$
= 0
$$
c) Show that: \( \text{curl curl} \, \mathbf{v} = \text{grad div} \, \mathbf{v} - \Delta \mathbf{v} \)

Again, it is necessary that \( \mathbf{v} \) is twice differentiable and second order derivatives can be interchanged in order. The latter is satisfied p.e. when \( \mathbf{v} \) is twice continuously differentiable (Schwarz’s theorem).

First, expand both \( \text{curl} \) operators into the coordinate form, then add a clever zero like \( + \frac{\partial^2}{\partial x \partial z} v_1 - \frac{\partial^2}{\partial x^2} v_1 \) in the \( x \)-coordinate to get \( \Delta v_1 \).

Note: Here \( \Delta \mathbf{v} \) is the vector Laplace operator applied to the vector field \( \mathbf{v} \). In the orthogonal Cartesian coordinates, \( \Delta \mathbf{v} \) simply returns a vector field equal to the scalar Laplacian applied to each vector component.

\[
\text{curl curl} \, \mathbf{v} = \text{curl} \left( \begin{array}{c}
\frac{\partial}{\partial y} v_3 - \frac{\partial}{\partial z} v_2 \\
\frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3 \\
\frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\frac{\partial}{\partial y} v_3 - \frac{\partial}{\partial z} v_2 \\
\frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3 \\
\frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1
\end{array} \right) \times \left( \begin{array}{c}
\frac{\partial^2}{\partial x \partial y} v_2 - \frac{\partial^2}{\partial y \partial z} v_1 \\
\frac{\partial^2}{\partial y \partial z} v_3 - \frac{\partial^2}{\partial z \partial x} v_2 \\
\frac{\partial^2}{\partial z \partial x} v_1 - \frac{\partial^2}{\partial x \partial y} v_3
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\frac{\partial^2}{\partial x \partial y} v_2 + \frac{\partial^2}{\partial y \partial z} v_3 - \frac{\partial^2}{\partial y \partial z} v_1 \\
\frac{\partial^2}{\partial y \partial z} v_3 + \frac{\partial^2}{\partial z \partial x} v_1 - \frac{\partial^2}{\partial x \partial z} v_2 \\
\frac{\partial^2}{\partial z \partial x} v_1 + \frac{\partial^2}{\partial x \partial y} v_3 - \frac{\partial^2}{\partial x \partial z} v_3
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\frac{\partial^2}{\partial x \partial y} v_2 + \frac{\partial^2}{\partial y \partial z} v_3 + \frac{\partial^2}{\partial z \partial x} v_1 - \Delta v_1 \\
\frac{\partial^2}{\partial y \partial z} v_3 + \frac{\partial^2}{\partial z \partial x} v_1 + \frac{\partial^2}{\partial x \partial y} v_3 - \Delta v_2 \\
\frac{\partial^2}{\partial z \partial x} v_1 + \frac{\partial^2}{\partial x \partial y} v_3 + \frac{\partial^2}{\partial x \partial z} v_3 - \Delta v_3
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3 \\
\frac{\partial}{\partial y} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3 \\
\frac{\partial}{\partial z} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3
\end{array} \right)
\]

\[
= \text{grad div} \, \mathbf{v} - \Delta \mathbf{v}
\]
d) Show that: \( \text{div} (v_1 \times v_2) = v_2 \cdot \text{curl} v_1 - v_1 \cdot \text{curl} v_2 \).

To define the \text{curl}'s, \( v_1, v_2 \) need to be (at least once) differentiable.

Let \( v_1(x, y, z) = \begin{pmatrix} v_{11}(x, y, z) \\ v_{12}(x, y, z) \\ v_{13}(x, y, z) \end{pmatrix} \) and \( v_2(x, y, z) = \begin{pmatrix} v_{21}(x, y, z) \\ v_{22}(x, y, z) \\ v_{23}(x, y, z) \end{pmatrix} \).

\[
\text{div} (v_1 \times v_2) = \text{div} \left( \begin{pmatrix} v_{12}v_{23} - v_{13}v_{22} \\ v_{13}v_{21} - v_{11}v_{23} \\ v_{11}v_{22} - v_{12}v_{21} \end{pmatrix} \right) \\
= \frac{\partial}{\partial x} \left( v_{12}v_{23} - v_{13}v_{22} \right) + \frac{\partial}{\partial y} \left( v_{13}v_{21} - v_{11}v_{23} \right) + \frac{\partial}{\partial z} \left( v_{11}v_{22} - v_{12}v_{21} \right) \\
= v_{12} \frac{\partial}{\partial x} v_{23} + v_{23} \frac{\partial}{\partial x} v_{12} - v_{13} \frac{\partial}{\partial x} v_{22} - v_{22} \frac{\partial}{\partial x} v_{13} \\
+ v_{21} \frac{\partial}{\partial y} v_{13} + v_{13} \frac{\partial}{\partial y} v_{21} - v_{11} \frac{\partial}{\partial y} v_{23} - v_{23} \frac{\partial}{\partial y} v_{11} \\
+ v_{22} \frac{\partial}{\partial z} v_{11} + v_{11} \frac{\partial}{\partial z} v_{22} - v_{12} \frac{\partial}{\partial z} v_{21} - v_{21} \frac{\partial}{\partial z} v_{12} \\
= \left( v_{21} \left( \frac{\partial}{\partial y} v_{13} - \frac{\partial}{\partial z} v_{12} \right) + v_{22} \left( \frac{\partial}{\partial z} v_{11} - \frac{\partial}{\partial x} v_{13} \right) + v_{23} \left( \frac{\partial}{\partial x} v_{12} - \frac{\partial}{\partial y} v_{11} \right) \right) \\
- \left( v_{11} \left( \frac{\partial}{\partial y} v_{23} - \frac{\partial}{\partial z} v_{22} \right) + v_{12} \left( \frac{\partial}{\partial z} v_{21} - \frac{\partial}{\partial x} v_{23} \right) + v_{13} \left( \frac{\partial}{\partial x} v_{22} - \frac{\partial}{\partial y} v_{21} \right) \right) \\
= v_2 \cdot \text{curl} v_1 - v_1 \cdot \text{curl} v_2
\]

**Aside: The Levi-Civita symbol**

Cross products may also be written with the Levi-Civita symbol or Epsilon tensor \( \epsilon_{ijk} \). It is defined as

\[
\epsilon_{ijk} = \begin{cases} 
1 & \text{if } ijk \text{ is an even permutation, e.g. 123} \\
-1 & \text{if } ijk \text{ is an odd permutation, e.g. 321} \\
0 & \text{if any indices repeat}
\end{cases}
\]

For example,

\[
\epsilon_{231} = -\epsilon_{132} = -(-\epsilon_{123}) = 1,
\]

so by switching indices, one can quickly arrive at a known value for \( \epsilon \). It can be generalized to \( n \) dimensions as

\[
\epsilon_{a_1a_2...a_n} = \prod_{1 \leq i < j \leq n} \text{sgn}(a_j - a_i),
\]

although the index switching described above might be faster for calculations.

With \( \epsilon_{ijk} \), a cross product of two vectors \( \vec{a}, \vec{b} \) can be written as

\[
(\vec{a} \times \vec{b})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a^j b^k,
\]
where $\vec{e}_i$ is the i’th orthogonal basis vector. Equipped with this knowledge, some identities involving the $\nabla$ operator are quite quick to prove:

Let’s take $\nabla \cdot (\nabla \times \vec{v})$ as an example:

$$\nabla \cdot (\nabla \times \vec{v}) = \partial_i \vec{e}_i (\epsilon_{ijk} \partial_j v_k \vec{e}_k)$$  \hspace{1cm} (4)

$$= \partial_i \epsilon_{ijk} \partial_j v_k \vec{e}_i \vec{e}_k$$  \hspace{1cm} (5)

$$= 0,$$  \hspace{1cm} (6)

as $\vec{e}_i \vec{e}_k = \delta_{ik}$, meaning that their scalar product equals 0 (they’re orthogonal!) if they are not the same two vectors. Now $\epsilon_{ijk}$ vanishes when two indices coincide while $\delta_{ik}$ is only non-zero if they do. Therefore, their product has to be zero. This is by the way a very general result: The product of any anti-symmetric tensor with a symmetric one is bound to be zero. In this example we have also used the Einstein summation convention, where a sum over repeated indices is implied.
Question 2: Rotation of a rigid body

Let us consider a rotating rigid body with rotation axis in the origin \( O \). Let the position vector be \( \mathbf{r} = (x, y, z) \) and the angular velocity \( \mathbf{\omega} = (\omega_1, \omega_2, \omega_3) \).

a) Angular velocity of \( \mathbf{\omega} = (\omega_1, \omega_2, \omega_3) \) implies by the right-hand rule that the velocity field \( \mathbf{v} \) of the rigid body is:

\[
\mathbf{v} = \mathbf{\omega} \times \mathbf{r} = \begin{pmatrix}
\omega_2 z - \omega_3 y \\
\omega_3 x - \omega_1 z \\
\omega_1 y - \omega_2 x
\end{pmatrix}
\]

b)

\[
\text{curl } \mathbf{v} = \text{curl } \begin{pmatrix}
\omega_2 z - \omega_3 y \\
\omega_3 x - \omega_1 z \\
\omega_1 y - \omega_2 x
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z) \\
\frac{\partial}{\partial x}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) \\
\frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial x}(\omega_2 z - \omega_3 y)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\omega_1 + \omega_1 \\
\omega_2 + \omega_2 \\
\omega_3 + \omega_3
\end{pmatrix}
\]

So \( \text{curl} \) actually measures the angular velocity. In the case of the rigid body, it is twice the angular velocity.

Moreover for the rotation of a rigid body, the \( \text{curl} \) of the velocity field which is itself a vector field, turns out to be constant in space.

Question 3: Flux in a Coulomb field

Consider an electric point charge \( e \) in the origin \( O \) of a cartesian coordinate system. Let \( \mathbf{v}(r) \) be the corresponding electric Coulomb field with

\[
\mathbf{v}(r) = C \frac{e}{r^3} \mathbf{r}
\]

with \( \mathbf{r} = (x, y, z)^T \) and \( r = \sqrt{x^2 + y^2 + z^2} \).

Then, the flux \( \phi \) of a point charge through a sphere with radius \( R \) and surface \( S \) is given by:

\[
\phi = \int_S C \frac{e}{R^3} \mathbf{r} \cdot \mathbf{n} dS
\]

With \( \mathbf{r} \) and \( \mathbf{n} \) parallel we get \( \mathbf{r} \cdot \mathbf{n} = R \) on the sphere. Hence,

\[
\phi = C \frac{e}{R^3} \int_S dS = C \frac{e}{R^2} 4\pi R^2 = 4\pi Ce
\]

So the flux is proportional to the charge \( e \) and independent of the radius.
Note that we may *not* use Gauss’ theorem, because $\mathbf{v}(\mathbf{r})$ is not continuously differentiable in the origin. In fact, $\mathbf{v}(\mathbf{r})$ is not continuous in the origin, hence it cannot be differentiable. However, off the origin we have

\[
\text{div } \mathbf{v}(\mathbf{r}) = Ce \left[ \frac{\partial x}{\partial x r^3} + \frac{\partial y}{\partial y r^3} + \frac{\partial z}{\partial z r^3} \right] \\
= Ce \left[ \frac{r^3 - x^2 r \cdot 2r}{r^6} + \frac{r^3 - y^2 r \cdot 2y}{r^6} + \frac{r^3 - z^2 r \cdot 2z}{r^6} \right] \\
= Ce \left[ \frac{1}{r^6} (3r^3 - 3(x^2 + y^2 + z^2) r) \right] \\
= 0
\]

So for any region $B$ not containing the origin with sufficiently smooth boundary $\partial B$ we conclude by Gauss’ theorem

\[
\int_{\partial B} Ce \mathbf{r} \cdot \mathbf{n} dS = \int_{\partial B} \mathbf{v}(\mathbf{r}) \cdot \mathbf{n} dS = \int_{B} \text{div } \mathbf{v}(\mathbf{r}) dV = \int_{B} 0 dV = 0.
\]

For example, this applies to the region obtained by removing an inner ball of radius $R_i$ from a ball of outer radius $R_o$. The flux through this region’s boundary is zero by Gauss’ theorem, and also zero because the influx at the inner surface ($4\pi Ce$ by the direct calculation) is exactly balanced by the outflux at the outer surface ($4\pi Ce$ as well).
Question 4: Potential fields

Let \( \mathbf{v} \) be a potential field with potential \( f \). This implies that

\[
\mathbf{v} = \nabla f
\]

a) In the self-test question, you showed that \( \text{curl} \, \nabla f = 0 \). Hence, it is easy to show that \( \mathbf{v} \) is vortex-free, namely:

\[
\text{curl} \, \mathbf{v} = \text{curl} \, \nabla f = 0
\]

This is stated in words as “Potential fields are vortex-free”.

b) Let \( \mathbf{v}(\mathbf{r}) \) be a Coulomb field with

\[
\mathbf{v}(\mathbf{r}) = -C \frac{\mathbf{r}}{r^3}
\]

with \( \mathbf{r} = (x, y, z) \) and \( r = \sqrt{x^2 + y^2 + z^2} \) in cartesian coordinates. So \( \mathbf{v} \) is also given by:

\[
\mathbf{v}(\mathbf{r}) = -C \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)^T
\]

Then

\[
\text{curl} \, \mathbf{v}(\mathbf{r}) = -C \text{curl} \left( \begin{pmatrix} \frac{x}{r^3} \\ \frac{y}{r^3} \\ \frac{z}{r^3} \end{pmatrix} \right)
\]

\[
= -C \left( \begin{array}{ccc}
\frac{\partial}{\partial y} \frac{z}{r^3} & -\frac{\partial}{\partial z} \frac{y}{r^3} \\
\frac{\partial}{\partial z} \frac{x}{r^3} & -\frac{\partial}{\partial x} \frac{z}{r^3} \\
\frac{\partial}{\partial x} \frac{y}{r^3} & -\frac{\partial}{\partial y} \frac{x}{r^3}
\end{array} \right)
\]

\[
= -C \left( \begin{pmatrix}
-\frac{3yz}{r^6} + \frac{y3z}{r^6} \\
-\frac{x3z}{r^6} + \frac{x3r}{r^6} \\
-\frac{y3r}{r^6} + \frac{y3x}{r^6}
\end{pmatrix}
\right)
\]

\[
= 0
\]