Question 1: Classifying PDE’s

a) Classify the following PDE’s in terms of their; order, linearity, coefficients and homogeneity:

- Second order, linear, varying co-efficient, in-homogenous
- First order, non-Linear, varying co-efficient, in-homogenous
- Third order, non-linear, quasi-linear, varying co-efficient, homogenous
- Second order, linear, constant co-efficient, non-homogenous

b) Classify the following second-order linear PDE’s in terms of their subclass (Elliptic, Hyperbolic or Parabolic):

i) Parabolic (Diffusion Equation) - (The example is changed to be simpler than the first PDE from part a), but the previous equation is also still parabolic)

In 2D the symmetrised matrix reads $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where first, second, third row and column refer to $x, y, t$ (and analogous in 3D). Hence the eigenvalues for $A$ are all positive (in the spatial dimension), except one that is zero (in the time dimension). This meets the first criteria for Parabolic, and we also have that the first derivative in time has non-zero co-efficient meeting the second condition.

ii) Hyperbolic

The eigenvalues for the spatial dimension’s are all $(\gamma + 5\eta)^2$ (Always positive, assuming the constants are real valued), and in the time dimension $-1$. Therefore we have a Hyperbolic equation.

iii) Elliptic - Poisson Equation

The eigenvalues of $A$ are all positive and one.

Question 2: Separation of Variables

a) Classify the above PDE.

The prefactor is a constant, hence we have a constant coefficient diffusion equation. It is second order, linear, constant co-efficient, homogenous, and can be subclassified to be parabolic.
b)  i) Sketch the initial condition. What do you think the solution profiles should look like?  

The initial condition is a simple square pulse from $x = 89$ to $x = 90$. Since we have a diffusion equation we would expect the solutions to spread smoothly in both directions, and then at long $t$ the domain to fill to a constant density.

ii) Let’s suppose a solution of the following form exists $C(x,t) = X(x)T(t)$. Rewrite the PDE using this substitution.

Making the substitution $C(x,t) = X(x)T(t)$ gives us,

\[
\begin{align*}
\frac{\partial(X(x)T(t))}{\partial t} &= \frac{1}{4} \frac{\partial^2(X(x)T(t))}{\partial x^2} \\
X(x)\frac{\partial T(t)}{\partial t} &= \frac{1}{4} \frac{\partial^2 X(x)}{\partial x^2}T(t)
\end{align*}
\]

and now since we have broken down our solution to function of single variables we can rewrite our partial derivatives as ordinary derivatives and so using dash notation,

\[
X(x)T'(t) = \frac{1}{4} X''(x)T'(t)
\]

iii) Show that we can rewrite this equation as

\[
\frac{X''(x)}{X(x)} = \frac{4T'(t)}{T(t)} = -\lambda^2
\]

where $\lambda$ is a constant. Why do we know it is equal to a constant?

First by straight re-arranging of above we have
Now we note that since \( x \) and \( t \) are independent variables, for this equation to hold both the LHS and RHS must be constant for all \( x \) and \( t \). We choose to call this constant \(-\lambda^2\), to make the later equations look nicer, and we get

\[
\frac{X''(x)}{X(x)} = \frac{4T'(t)}{T(t)} = -\lambda^2
\]  

(11)

iv) Identify the two ODE’s (for \( X(x) \) and \( T(t) \)) and their boundary conditions and solve them. (Hint: See pg. 78 for the general solutions for these ODE’s). Apply the boundary conditions for the equation of \( X(x) \), this should give you an equation relating to \( \lambda \).

Our two ODE’s are

\[
X''(x) - X(x)\lambda^2 = 0 \tag{12}
\]

\[
4T'(t) - T(t)\lambda^2 = 0 \tag{13}
\]

We will not step through the solution to these two ODE’s here, (Both are constant co-efficient linear equations and can be solved by supposing solutions of the form \( y = e^{ri x} \) for \( i = 1..d \), where \( d \) is the order of equation, among other methods.)

We have the general solutions

\[
X(x) = B_1 \sin \lambda x + B_2 \cos \lambda x \tag{14}
\]

\[
T(t) = A e^{-\lambda^2 t} \tag{15}
\]

Now we can apply our boundary conditions above for \( X(x) \), so start with

\[
X'(x) = B_1 \cos \lambda x - B_2 \sin \lambda x \tag{16}
\]

so now applying \( X'(0) = 0 \) gives us that \( B_1 = 0 \), and now our second condition \( X'(199) = 0 \) gives us

\[
B_2 \sin \lambda 199 = 0 \tag{17}
\]

Now again we want non-trivial solutions (\( B_2 \) cannot therefore be zero) so want the solutions to

\[
\sin (\lambda 199) = 0. \tag{18}
\]
\( \sin x = 0 \) has solutions \( x = n\pi \) where \( n \in \mathbb{Z} \) is an integer. Therefore, this restricts our constant to \( \lambda = \frac{n\pi}{199} \). So now we have

\[
X(x) = B_2 \cos \frac{n\pi x}{199} \\
T(t) = A e^{-t\left(\frac{n\pi}{199}\right)^2}
\]

where \( n \) is an integer, this is we have a infinite set of solutions to the problem.

v) What is the superposition principle? The superposition principle applies to linear homogeneous PDE’s (and also ODE’s) and tells us that if we have a set of solutions \( y_i(x) \) to the equation then any linear combinations of this set are also solutions to this equation.

Using the superposition principle for our solutions to \( X(x), T(t) \) (each expressed in terms of \( n, n \in \mathbb{N} \)) we can write

\[
C(x, t) = \sum_{n=0}^\infty B_{2,n} \cos \frac{n\pi x}{199} \cdot \tilde{A}_n e^{-t\left(\frac{n\pi}{199}\right)^2}
\]

with unknown coefficients \( B_{2,n}, \tilde{A}_n, n \in \mathbb{N} \). We need only consider positive integers here as our solution is even with regards to \( n \).

Rewrite as

\[
C(x, t) = \sum_{n=0}^\infty A_n \cos\left(\frac{n\pi x}{199}\right) e^{-t\left(\frac{n\pi}{199}\right)^2}
\]

where we have combined the constants \( B_{2,n} \) and \( \tilde{A}_n \) to \( A_n \).

vi) We still need to satisfy the initial condition, namely

\[
C(x, 0) = \sum_{n=0}^\infty A_n \cos\left(\frac{n\pi x}{199}\right).
\]

This is just the cosine series representation of our initial condition and it can be used to show that our particular solution to the problem is

\[
C(x, t) = \frac{20}{199} - 2 \sum_{n=1}^\infty \left( \frac{\sin\left(\frac{89n\pi}{199}\right) - \sin\left(\frac{109n\pi}{199}\right)}{n\pi} \right) \cos\left(\frac{n\pi x}{199}\right) e^{-t\left(\frac{n\pi}{199}\right)^2}.
\]

You are not expected to be able to do this step but see http://en.wikipedia.org/wiki/Separation_of_variables

You can solve the boundary condition by multiplying both sides by \( \cos\left(\frac{m\pi x}{199}\right) \) and integrating across your domain, where \( m \) is also a positive integer. Orthogonality will then reduce this to a solvable integral.

Note that our solution now includes an infinite sum!, write some code (Matlab?) to plot this solution for different times, what impact does the number of terms in your
sum make?.

Here you should see that for a low number of terms the solution struggles to provide a reasonable solution (Introduces oscillations), and for the initial condition struggles to resolve the steep front, as the solution smooths out less terms are required. There is also a reasonable computational cost to calculating the series solution with a reasonable number of terms.

vii) On an infinite domain we can solve the same problem using similarity solutions to give the following solution

\[ C(x, t) = \frac{1}{2} \left( \text{erf}\left( \frac{x - 89}{\sqrt{t}} \right) - \text{erf}\left( \frac{x - 109}{\sqrt{t}} \right) \right) \]  \hspace{1cm} (25)

compare this solution at different times to our separation of variables solution.

We see that the solutions are indistinguishable for smaller \( t \), but as the solution becomes larger and the boundary comes into effect the solutions diverge, as the similarity solution is over a infinite domain.
Figure 3: Separation of variables solution vs. Similarity solution: $N = 100, t = 50$

Figure 4: Separation of variables solution vs. Similarity solution: $N = 100, t = 5000$