Burgers’ Equation with Adaptive Particles

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1 Equations

1.1 1D

The equation to be solved is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{R} \frac{\partial^2 u}{\partial x^2},
\]

\[
u(x, t = 0) = u_0(x).
\]

where \(R\) is Reynold’s number and \(u(x, t) \in \mathbb{R}\) is the velocity. We carry \(u\) on particles which move with

\[
\frac{dx}{dt} = u + w,
\]

where \(w(x, t) \in \mathbb{R}\) is some arbitrary particle velocity component resulting from particle management. We can rewrite (1) as

\[
\frac{\partial u}{\partial t} + (u + w) \frac{\partial u}{\partial x} = \frac{1}{R} \frac{\partial^2 u}{\partial x^2} + w \frac{\partial u}{\partial x} \quad \text{or}
\]

\[
Du = \frac{1}{R} \frac{\partial^2 u}{\partial x^2} + w \frac{\partial u}{\partial x}.
\]

2 Stability of the discretized equations

2.1 1D, Explicit Euler time stepping, central second-order FD in space (FTCS)

We have

\[
x^{n+1}_i = x^n_i + \Delta t (u^n_i + w^n_i)
\]

plus remeshing and

\[
u^{n+1}_i = u^n_i + \Delta t \left\{ \frac{1}{R} \frac{u^{n+1}_{i+1} - 2u^n_i + u^{n-1}_{i-1}}{h^2} + w^n_i \frac{u^n_{i+1} - u^n_{i-1}}{2h} \right\}.
\]

We now take one mode of \(u\): \(u_k(x^n_i, t^n) = \hat{u}^n_k e^{ikh}, x_{i+1} = x_i + h, \) and evaluate the gain in \(u_k\) as

\[
\left| \hat{u}^{n+1}_k \right| = 1 + \Delta t \left\{ \frac{e^{ikh} - 2 + e^{-ikh}}{R^2} + w^n_i \frac{e^{ikh} - e^{-ikh}}{2h} \right\}
\]

\[
= 1 + \Delta t \left\{ -\frac{2}{Rh^2} + \frac{2 \cosh(ikh)}{Rh^2} + w^n_i \frac{\sinh(ikh)}{h} \right\}
\]

\[
= 1 + \frac{2\Delta t}{Rh^2} \left\{ -1 + \cos(kh) + i \frac{Rh w^n_i}{2} \sin(kh) \right\}.
\]
The gain must not be greater than one, thus,
\[
\frac{\hat{u}_{k}^{n+1}}{\hat{u}_{k}^{n}} = 1 + \frac{2\Delta t}{Rh^2} \{ \cos(kh) - 1 \}^2 + \left[ \frac{\Delta t w_i^n}{h} \sin(kh) \right]^2
\]
\[
= 1 + \frac{4\Delta t}{Rh^2} \{ \cos(kh) - 1 \} + \frac{4\Delta t^2}{Rh^2} \{ \cos(kh) - 1 \}^2 + \frac{\Delta t^2 w_i^{n+2}}{h^2} \{ 1 - \cos^2(kh) \} \quad \text{for} \quad \Delta t, R, h > 0 \]
\[
\Delta t < \frac{Rh^2 [1 - \cos(kh)]}{1 - \frac{R^2 h^2 w_i^{n+2}}{4}} = Rh^2 G_S(a)
\]
where
\[
G_S(a) = \frac{1 - a}{[1 - S] a^2 - 2a + 1 + S} = \frac{1}{(S - 1)a + 1 + S}
\]
with \( S = \frac{R^2 h^2 w_i^{n+2}}{4} > 0 \) and \( a \in [-1, 1] \). To be on the safe side, we have to find \( \min_a G_S(a) \) and enter this into the time step limit (9). We compute

\[
G_S(-1) = \frac{1}{2}, \quad G_S(1) = \frac{1}{2S}, \quad \frac{dG_S}{da} = \frac{1 - S}{[(1 - S)a - 1 - S]^2}.
\]

Thus, there are no extrema and \( G_S \) is monotonically increasing if \( S < 1 \), constant for \( S = 1 \), and decreasing else (note that the pole of \( G_S \), at \( a = (1 + S)/(1 - S) \), is not within the interval \([-1, 1]\) for \( S > 0 \)). This gives us

\[
\Delta t < \frac{Rh^2}{2}, \quad \text{if} \quad S \leq 1,
\]
\[
\Delta t < \frac{2}{R \epsilon \eta^2}, \quad \text{else}.
\]

### 2.2 1D, Explicit Euler time stepping, DC PSE operators (FTCS)

We have
\[
x_q^{n+1} = x_q^n + \Delta t (u_q^n + w_q^n)
\]
and – with implicit volumes contained in \( \eta^q \) via DC –
\[
u_q^{n+1} = u_q^n + \Delta t \left\{ \frac{1}{R \epsilon^2} \sum_p (u_p^n - u_q^n) \eta^q \eta_\epsilon^{(2)}(x_q^n - x_p^n) + \frac{w_i^n}{\epsilon} \sum_p (u_p^n + u_q^n) \frac{x_q^n - x_p^n}{\epsilon} \eta^q \eta_\epsilon^{(2)}(x_q^n - x_p^n) \right\}
\]
\[
= u_q^n + \Delta t \sum_p \left\{ \frac{1}{R \epsilon^2} (u_p^n - u_q^n) + \frac{w_i^n}{\epsilon} (u_p^n + u_q^n) \frac{x_q^n - x_p^n}{\epsilon} \right\} \eta^q \eta_\epsilon^{(2)}(x_q^n - x_p^n).
\]

We now take one mode of \( w \): \( u_k(x_q^n, t^n) = \hat{u}_k e^{ikx_q} \) and evaluate the gain in \( u_k \) as
\[
\frac{\hat{u}_k^{n+1}}{\hat{u}_k^n} |_{q} = 1 + \Delta t \sum_p \left\{ \frac{1}{R \epsilon^2} (e^{-ik(x_q^n - x_p^n)} - 1) + \frac{w_i^n}{\epsilon} (e^{-ik(x_q^n - x_p^n)} + 1) \frac{x_q^n - x_p^n}{\epsilon} \right\} \eta^q \eta_\epsilon^{(2)}(x_q^n - x_p^n)
\]
\[
= 1 + \frac{\Delta t}{R \epsilon^2} \left\{ \hat{\eta}^q, \eta_\epsilon^{(2)}(k \epsilon) - Z_h^{q,0,n} + w_i^n R \epsilon Z_h^{q,1,n} + i w_i^n R \epsilon \frac{d \hat{\eta}^q, \eta_\epsilon^{(2)}(k \epsilon)}{d(k \epsilon)} \right\}
\]
where
\[ \eta^{q,(2),n}(k \varepsilon) = \sum_p e^{-ik(x^n_q - x^n_p)} \eta^{q,(2)}(x^n_q - x^n_p), \]  
(17)
\[ Z^{q,\alpha,n}_h = \sum_p \frac{(x^n_q - x^n_p)^\alpha}{\varepsilon |\alpha|} \eta^{q,(2)}(x^n_q - x^n_p). \]  
(18)

The gain must not be greater than one, thus,
\[ \left| \frac{\hat{u}^n_{k+1}}{\hat{u}^n_k} \right|_q^2 = 1 + \frac{2 \Delta t}{R \varepsilon^2} \text{Re}(K) + \frac{\Delta t^2}{R^2 \varepsilon^4} |K|^2 \]  
\[ \Delta t \| < \frac{2 R \varepsilon^2 \text{Re}(K)}{|K|^2 - \lambda^{*}_{\text{diff}}} \]  
\[ = (-\lambda^{*}_{\text{diff}} + O(w R \varepsilon)) R h^2 \]  
(19)

for \( k \varepsilon \in (0, \pi/c] \) where \( K = \eta^{q,(2),n}(k \varepsilon) - Z^{q,0,n}_h + O(w R \varepsilon) \), thus, the real part of \( K \) needs to be negative. The \( \lambda^{*}_{\text{diff}} \) is from our DC paper, thus, as long as \( w R \varepsilon \ll 1 \), the time step limit is as for the pure diffusion equation.