Exercise 5
Release: 11.11.2019
Due: 18.11.2019

Question 1: Least-Squares formulation, Normal equation

a) Given three points $P_i = (x_i, y_i), \ i = 1, 2, 3$, wherein

$$\begin{array}{c|ccc}
  x_i & 0 & 1 & 2 \\
  y_i & 5.41 & 5.17 & 5.93 \\
\end{array}$$

Determine a linear function $y = f(x) = ax + b$, so that the sum of the error squares in the y-direction

$$\sum_{i=1}^{3} |f(x_i) - y_i|^2$$

is minimized

b) Consider the matrix $A (3 \times 2)$ and vector $b (3 \times 1)$, given by

$$A = \begin{pmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

i) Write the least-square form which leads to the solution of linear system $Ax = b$.

ii) Check if the method of Normal Equations is stable for matrix $A$ and $0 < \epsilon \ll 1$.

Question 2: Least-squares, QR decomposition and SVD

Following are the velocity measurements $f(t)$ in $ms^{-1}$ from the pitot-tube of a descending airplane at time $t_i, \ i = 1, 2, ..., 10$.

$$\begin{array}{c|cccccccccc}
  t_i & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1.0 \\
  f_i & 100 & 34 & 17 & 12 & 9 & 6 & 5 & 4 & 4 & 2 \\
\end{array}$$

We express the unknown function $f(t) = \sum_{j=1}^{4} \lambda_j \phi_j(t)$ as the linear combination of known functions $\phi_j(t), \ j = 1, 2, 3, 4$, given by,

$$\phi_1(t) = \frac{1}{t}, \ \phi_2(t) = \frac{1}{t^2}, \ \phi_3(t) = e^{-(t-1)}, \ \phi_4(t) = e^{-2(t-1)}$$

Determine the coefficients $\lambda_j$, the linear combination such that

$$\sum_{i=1}^{10} |f(t_i) - f_i|^2$$

is minimized
Question 3: SVD decomposition by hand
Given the matrices A and B,

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

a) Find the rank and singular values of the matrix A and matrix B.

b) Find the SVD decomposition of the matrix A and B. Also comment on the uniqueness of the decomposition.