Exercise 10
Release: 13.12.2018
Due: 20.12.2018

Question 1: Derivative approximation
Given the function \( f(x) = e^{-x^2} \), sample the function at \( N \) uniformly spaced grid points in the interval \([-1, 1]\). Compute the derivative \( f'(x_i) \) and the second derivative \( f''(x) \) of the function at sample point \( x_i, i = 1, 2, \ldots, N - 1 \) and spacing \( h = \frac{2}{N} \) using

a) central difference \( f'(x) = \frac{f(x+h) - f(x-h)}{2h} \) for varying \( N \)

b) forward difference \( f'(x) = \frac{f(x+h) - f(x)}{h} \) for varying \( N \)

Compute the error of approximation w.r.t the analytical solution \( f'(x) = -2xe^{-x^2} \) for varying \( N \) for both a) and b). Plot the error for varying \( N \)

Question 2: Time integration
Given the initial value problem

\[ \frac{\dot{y}}{f(t, y(t))}, \ y(t_0) = y_0 \]

a) The local truncation error of a time integration scheme is the numerical error made in a single step. Whereas, the global truncation error is the accumulated numerical error for computing the solution upto time \( t \) starting from \( t_0 \). Derive the expressions for the local and global truncation errors of the Euler method.

b) For the differential equation \( \dot{y} = -100y \) and the initial condition \( y(0) = 1 \), evaluate the function at discrete times using explicit-Euler at \( t = 0.01, 0.05, 0.1 \) and compare the results with analytical solution. Compute both the local and global error of the time-stepping scheme for the same.

c) Compute the 4th order Central difference for a \( C^5 \) function \( f \) and prove that its truncation error is

\[ E(f, h) = \frac{h^4 f^{(5)}(c)}{30} \]

Question 3: Programming Task
Consider the damped harmonic oscillator

\[ \ddot{x}(t) + 0.5\dot{x}(t) + x(t) = 0 \]

with initial conditions \( x(0) = 1 \) and \( \dot{x}(0) = 0 \).
a) Create a Matlab/Python subroutine to solve the given equation using explicit Euler method with spacing $h$ as a parameter.

b) Using a) Plot the error for different step sizes $h$ where the error is computed from the analytical solution given below

\[ x(t) = \frac{e^{-\frac{t}{4}}}{15} \left( 15 \cos \left( \frac{\sqrt{15} t}{4} \right) + \sqrt{15} \sin \left( \frac{\sqrt{15} t}{4} \right) \right) \]