**Solution 12**

Release: 13.01.2020  
Due: 20.01.2020

**Question 1: Trapezoidal rule and Implicit time-stepping**

Consider the Initial value problem (IVP)
\[
dot{y} = f(y), \quad y(0) = y_0
\]

and the trapezoidal method
\[
\tilde{y}_{j+1} = \tilde{y}_j + \frac{h}{2} \left( f(\tilde{y}_j) + f(\tilde{y}_{j+1}) \right); \quad \tilde{y}_0 = y(0)
\]

To calculate the value \(\tilde{y}_{j+1}\), select the starting value of \(\tilde{y}_0 = y(0)\) for the fixed-point iteration
\[
\tilde{y}_k^{j+1} = \tilde{y}_j + \frac{h}{2} \left( f(\tilde{y}_j) + f(\tilde{y}_k^{j+1}) \right), \quad k = 0, 1, 2, \ldots
\]

Show that the sequence \(\tilde{y}_k^{j+1}\) converges for \(k \to \infty\) to the fixed point \(\tilde{y}_{j+1}\), if \(h\) is small enough and \(f(y)\) does not vary much in the chosen interval.

**Solution:**

According to fixed-point theorem, we need to show that the function \(g(y)\) is contraction map and is lipschitz continuous,
\[
y = g(y) = \tilde{y}_j + \frac{h}{2} \left( f(\tilde{y}_j) + f(y) \right)
\]

For the function \(g(y)\) to be a contraction map, we need in the interval \(I = [\tilde{y}_j - r, \tilde{y}_j + r]\),
\[
|y - \tilde{y}_j| = \frac{h}{2} \left( f(\tilde{y}_j) + f(y) \right) \leq h \max_{y \in I} f(y) \leq r
\]

Let's assume a bound on \(f(y)\) i.e. \(|f(y)| \leq M_1 \implies M_1 h \leq r\) (*)

For the function \(g(y)\) to be lipschitz continuous in the interval \(I\)
\[
|g(y) - g(y')| = \left| \tilde{y}_j + \frac{h}{2} \left( f(\tilde{y}_j) + f(y) \right) - \tilde{y}_j + \frac{h}{2} \left( f(\tilde{y}_j) + f(y') \right) \right|
\]
\[
= \frac{h}{2} |f(y) - f(y')|
\]
\[
= \frac{h}{2} |(y - y') f'(\eta)|
\]
such that $y < \eta < y'$. We then can put a bound on the derivative $f'$, i.e. $|f'(\eta)| < M_2$.

$$|g(y) - g(y')| \leq \frac{h}{2} M_2 |(y - y')|$$

So the function $g(y)$ is a contraction map, given $\frac{h}{2} M_2 < 1 \implies h < \frac{2}{M_2}$ (**).

Combining the two bounds (*) and(**), i.e.

$$h < \frac{2}{M_2}, \quad r \geq h M_1$$

and for any initial guess $y_0 \in [\bar{y}_j - r, \bar{y}_j + r]$, we will converge to the fixed-point $\bar{y}_{j+1}$ according to the fixed-point theorem. By assuming constraints on the function (*) and its derivative (**), we showed that the update rule converges to the fixed-point.

**Question 2: Fixed-point and Implicit schemes**

Given the differential equation of the damped harmonic oscillator

$$\ddot{x} + 0.5 \dot{x} + x = 0$$

with the initial conditions $x(0) = 1, \dot{x}(0) = 0$

a) Convert the second order differential equation to a system of first order Differential equations.

Make the substitution $x_1 = x$ and $x_2 = \dot{x}$, we end with the system of equations

$$\begin{align*}
\dot{x}_1 & = x_2 \\
\dot{x}_2 & = -0.5 x_2 - x_1
\end{align*}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 0$

b) Approximate $x(h)$ using trapezoidal method for $h = 0.5$

**Solution:**

The trapezoidal rule (semi-implicit) reads,

$$\bar{x}_{j+1} = \bar{x}_j + \frac{h}{2} (f(t_j, \bar{x}_j) + f(t_{j+1}, \bar{x}_{j+1}))$$

$$x(h) = x_1(h) \approx x_1^{(1)}$$

$$\begin{pmatrix}
\bar{x}_1^{(1)} \\
\bar{x}_2^{(1)}
\end{pmatrix} = \begin{pmatrix}
\bar{x}_1(0) \\
\bar{x}_2(0)
\end{pmatrix} + \frac{h}{2} \begin{pmatrix}
f_1(0, \bar{x}_1(0), \bar{x}_2(0)) + f_1(h, \bar{x}_1^{(1)}, \bar{x}_2^{(1)}) \\
f_2(0, \bar{x}_1(0), \bar{x}_2(0)) + f_2(h, \bar{x}_1^{(1)}, \bar{x}_2^{(1)})
\end{pmatrix}$$

making the substitutions for initial conditions and $h$, we get

$$\begin{align*}
\bar{x}_1^{(1)} & = 1 + \frac{h}{2} (0 + \bar{x}_2^{(1)}) \\
\bar{x}_2^{(1)} & = 0 + \frac{h}{2} (-1 - 0.5 \bar{x}_2^{(1)} - \bar{x}_1^{(1)}) \\
\bar{x}_2^{(1)} & = -\frac{h}{2} - \frac{h}{4} \bar{x}_2^{(1)} - \frac{h}{2} - \frac{h^2}{4} \bar{x}_2^{(1)} \\
\bar{x}_2^{(1)} & = -\frac{h}{1 + \frac{h}{4} + \frac{h^2}{4}} = -0.42105
\end{align*}$$
Substituting for $\tilde{x}_2^{(1)}$, we get

$$x(h) = x_1(h) \approx \tilde{x}_1^{(1)} = 1 + \frac{h}{2} \tilde{x}_2^{(1)} = 0.894737$$

**Question 3: Butcher tableau**

Given the Butcher Tableau of the $\vartheta$-procedure

<table>
<thead>
<tr>
<th>$\vartheta$</th>
<th>$\vartheta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

a) for which $\vartheta$ is the procedure explicit or implicit?

**Solution:**

$$k_1 = f(t_j + \vartheta h, \tilde{x}_j + h \vartheta k_1)$$

$$\tilde{x}_{j+1} = \tilde{x}_j + h k_1$$

From the above formula, it is clear that for $\vartheta = 0$ is explicit and $\vartheta \neq 0$ is implicit.

b) determine the order of error of the procedure depending on the $\vartheta$.

$$f(t + \vartheta h, \bar{x} + h \vartheta k_1) = f(t, x) + \vartheta h f_i(t, x) + h \vartheta k_1 f_x(t, x) + \frac{\vartheta^2 h^2}{2} f_{tt}(t, x)$$

$$+ \frac{\vartheta^2 h^2 k_1^2}{2} f_{xx}(t, x) + \frac{\vartheta^2 h^2 k_1}{2} f_{tx}(t, x) + O(h^3)$$

Substituting the above in the update formulate $x(t + h) = x(t) + h k_1$, we get

$$x(t + h) = x(t) + h f(t, x) + \vartheta h^2 f_i(t, x) + h^2 \vartheta k_1 f_x(t, x) + \frac{\vartheta^2 h^3}{2} f_{tt}(t, x)$$

$$+ \frac{\vartheta^2 h^3 k_1^2}{2} f_{xx}(t, x) + \frac{\vartheta^2 h^3 k_1}{2} f_{tx}(t, x) + O(h^4)$$

after grouping, truncating the 3rd order terms and for substituting $k_1 = f(t, x)$, we get

$$x(t + h) = x(t) + h f(t, x(t)) + h^2 \vartheta [f_i(t, x) + f_x(t, x)f(t, x)] + O(h^3) \quad (\ast)$$

$$x(t + h) = x(t) + h \dot{x}(t) + \frac{h^2}{2} \ddot{x}(t) + O(h^3) \quad (\ast\ast)$$

Comparing (\ast) and (\ast\ast), we can infer that for $\vartheta = \frac{1}{2}$, the method is $O(h^2)$, else $O(h)$
c) sketch the stability area for the methods with \( \vartheta = 0, \frac{1}{2}, 1 \).

Let us take the model problem

\[
\dot{x} = \lambda x, \quad x(0) = x_0, \quad \lambda \in \mathbb{C}
\]

For \( \vartheta = 0 \), the update rule is

\[
\tilde{x}_{j+1} = \tilde{x}_j + hf(t_j, \tilde{x}_j) \\
= \tilde{x}_j + \lambda \tilde{x}_j \\
= \tilde{x}_j(1 + \lambda)
\]

\( \mu = h\lambda \implies R(\mu) = 1 + \mu \). The update is stable if \( |R(\mu)| < 1 \).

Since \( \mu \in \mathbb{C} \implies \mu = u + iv \implies |R(\mu)|^2 = (1 + u)^2 + v^2 \).

Solving for \( |R(\mu)|^2 = (1 + u)^2 + v^2 < 1 \), we get the stability regions.

For \( \vartheta = \frac{1}{2} \), the update rule is

\[
\tilde{x}_{j+1} = \tilde{x}_j + hf(t_j + \frac{h}{2}, \frac{\tilde{x}_j + \tilde{x}_{j+1}}{2}) \\
= \tilde{x}_j + h\lambda \left( \frac{1}{2} \tilde{x}_j + \frac{1}{2} \tilde{x}_{j+1} \right) \\
\tilde{x}_{j+1} \left( 1 - \frac{1}{2} h\lambda \right) = \tilde{x}_j \left( 1 + \frac{1}{2} h\lambda \right)
\]

\( \mu = h\lambda \implies R(\mu) = \frac{2+\mu}{2-\mu} \implies |R(\mu)|^2 = \frac{(2+u)^2 + v^2}{(2-u)^2 + v^2} = 1 \implies u = 0 \).

For \( \vartheta = 1 \), the update rule is

\[
\tilde{x}_{j+1} = \tilde{x}_j + hf(t_j + h, \tilde{x}_{j+1}) \\
= \tilde{x}_j + h\lambda \tilde{x}_{j+1} \\
\tilde{x}_{j+1} \left( 1 - h\lambda \right) = \tilde{x}_j \implies R(\mu) = \frac{1}{1 - \mu}
\]

For \( \mu = u + iv \implies |R(\mu)| = \frac{1}{(1-u)^2 + v^2} \). Solving for \( |R(\mu)|^2 = \frac{1}{(1-u)^2 + v^2} < 1 \), we get the stability regions.
Question 4: Programming Task

a) Implement the 3-step Adams-Bashforth procedure

\[ \tilde{x}^{n+1} = \tilde{x}^n + \frac{h}{12} \left( 23f^n - 16f^{n-1} + 5f^{n-2} \right) \]

Assume that the 3 starting values \( x^0, x^1, x^2 \) are known.

b) apply your program to

\[ \dot{x}_1 = bx_1 - cx_1x_2 \]
\[ \dot{x}_2 = -dx_2 + cx_1x_2 \]

for \( b = 1, d = 10, c = 1 \), initial conditions \( x_1(0) = \frac{1}{2}, x_2(0) = 1 \) and \( t_f = 10 \). The starting values are, for \( h = \frac{1}{100} \),

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x_1(t) )</th>
<th>( x_2(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.5000000000000000</td>
<td>1.0000000000000000</td>
</tr>
<tr>
<td>0.01</td>
<td>0.50023020652423</td>
<td>0.90937367706190</td>
</tr>
<tr>
<td>0.02</td>
<td>0.50089337004375</td>
<td>0.82696413439848</td>
</tr>
</tbody>
</table>

The above system of ODEs represents the predator-prey model, where \( x_1 \) is the prey and \( x_2 \) is the predator concentrations.

c) plot the two populations \( x_1, x_2 \) as a function of time \( t \) and also the trajectory \( x_1(t), x_2(t) \) in the \((x_1, x_2)\) plane (phase plane).