Solution 13

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Question 1: Stability

Consider the Linear Differential Equation system

\[ \dot{x} = \begin{pmatrix} -1001 & 999 \\ 999 & -1001 \end{pmatrix} x \]

Solution:

\[ \dot{x} = Ax = TDT^{-1}x \]
\[ \Rightarrow T^{-1}\dot{x} = DT^{-1}x, \quad y = T^{-1}x \]
\[ \dot{y} = Dy \quad \text{where, } D = \text{diag}(\lambda_i) \]

Eigen-values of \( A \):

\[ \lambda_1 = -2000; \quad v_1 = (-1, 1)^T \]
\[ \lambda_2 = -2; \quad v_2 = (1, 1)^T \]

\[ y(t) = \begin{pmatrix} y_1(0)e^{-\lambda_1 t} \\ y_2(0)e^{-\lambda_2 t} \end{pmatrix}; \quad x(t) = Ty(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \]

\[ x(t) = \begin{pmatrix} -y_1(0)e^{-\lambda_1 t} + y_2(0)e^{-\lambda_2 t} \\ y_1(0)e^{-\lambda_1 t} + y_2(0)e^{-\lambda_2 t} \end{pmatrix}; \quad x(0) = \begin{pmatrix} -y_1(0) + y_2(0) \\ y_1(0) + y_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \]

\[ \Rightarrow y_2(0) = \frac{1}{2} (x_1(0) + x_2(0)); \quad y_1(0) = \frac{1}{2} (x_2(0) - x_1(0)). \]

\[ x(t) = \begin{pmatrix} \frac{1}{2} (x_1(0) - x_2(0)) e^{-2000t} + \frac{1}{2} (x_1(0) + x_2(0)) e^{-2t} \\ \frac{1}{2} (x_2(0) - x_1(0)) e^{-2000t} + \frac{1}{2} (x_1(0) + x_2(0)) e^{-2t} \end{pmatrix} \]

a) we want to apply the explicit Euler procedure to this System. How big must the step size be for the numerical solution to the starting conditions

i) \( x(0) = (-1, 1)^T \)

the analytical solution is then,

\[ x(t) = \begin{pmatrix} \frac{1}{2} (-2) e^{-2000t} \\ \frac{1}{2} (2) e^{-2000t} \end{pmatrix} \]
So the only eigen-value left is \( \lambda_1 = -2000 \). For stability of explicit schemes, we need \( h\lambda \in (-2, 0) \),

\[
-2000h \in (-2, 0) \\
h < 0.001
\]

ii) \( x(0) = (1, 1)^T \)
the analytical solution is then,

\[
x(t) = \begin{pmatrix} \frac{1}{2} (2) e^{-2t} \\ \frac{1}{2} (2) e^{-2t} \end{pmatrix}
\]

So the only eigen-value left is \( \lambda_2 = -2 \). For stability of explicit schemes, we need \( h\lambda \in (-2, 0) \),

\[
-2h \in (-2, 0) \\
h < 1
\]

iii) \( x(0) = (2, 0)^T \)
the analytical solution is then,

\[
x(t) = \begin{pmatrix} \frac{1}{2} e^{-2000t} + \frac{1}{2} e^{-2t} \\ -\frac{1}{2} e^{-2000t} + \frac{1}{2} e^{-2t} \end{pmatrix}
\]

Here we choose the largest eigen-value \( \lambda_1 = -2000 \). For stability of explicit schemes, we need \( h\lambda \in (-2, 0) \),

\[
-2000h \in (-2, 0) \\
h < 0.001
\]

for stable qualitative behaviours.

**Question 2: Stiff Differential Equations**

Consider the IVP,

\[
y'(t) = -15y(t)
\]

with the initial condition \( y(0) = 1 \) and \( t \geq 0 \).

a) Find the exact analytical solution to the problem.

b) Use explicit Euler scheme to solve the problem with step size \( h = \frac{1}{4} \) and \( h = \frac{1}{8} \)

c) Use the implicit Trapezoidal scheme to solve the problem and plot a graph comparing explicit Euler (\( h = \frac{1}{4}, \frac{1}{8} \)), implicit Trapezoidal (\( h = \frac{1}{8} \)), and the analytical solution.

**Solution**

The exact solution is

\[
y(t) = e^{-15t} \text{ with } y(t) \to 0 \text{ as } t \to \infty
\]
Question 3: Classification of PDEs

Classify the following PDEs with respect to their order and type (parabolic, hyperbolic and elliptic)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D \Delta u, \\
\Delta u &= 0, \\
u_{tt} - c^2 u_{xx} &= 0, \\
u_{xx} + xu_{yy} &= 0.
\end{align*}
\]

Solution:

The general case of second-order linear partial differential equation (PDE) with two independent variables is given by

\[
A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G.
\]

We base our classification on the sign of the quantity \(B^2 - 4AC\).

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D \Delta u, \text{ parabolic} \\
\Delta u &= 0 \text{ elliptic} \\
u_{tt} - c^2 u_{xx} &= 0 \text{ Hyperbolic} \\
u_{xx} + xu_{yy} &= 0.
\end{align*}
\]

\((B^2 - 4AC) = -4x \implies \) for \(x > 0\) elliptic, \(x = 0\) parabolic, \(x < 0\) hyperbolic