**Question 1: Solution of PDEs**

Consider the PDE for advection equation \( u_t + cu_x = 0 \). Assuming that, we are only allowed to Fourier transform along \( x \), i.e.

\[
\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\xi x} dx \quad (*)
\]

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{i\xi x} dx \quad (**)\]

(a) Formulate the analytical solution for \( u(x, t) \) given the initial data for \( \hat{u}(\xi, 0) \).

\[
u_t + cu_x = 0 \quad \text{(Multiply by } e^{-i\xi x} \text{ and integrate from } -\infty \text{ to } \infty)
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u_t + cu_x) e^{-i\xi x} dx = 0
\]

\[
\hat{u}_t(\xi, t) + ci\xi \hat{u}(\xi, t) = 0
\]

\[
\hat{u}_t = -i\xi c \hat{u} \quad \text{This is a ODE of the from } \dot{y} = \lambda y
\]

The initial condition i.e.

\[
\hat{u}(\xi, 0) = \hat{\eta}(\xi) = \int_{-\infty}^{\infty} \eta(x) e^{-i\xi x} dx
\]

So, the analytical solution for \( \hat{u}_t \) for each \( \xi \) and initial condition \( \hat{u}(\xi, 0) = \hat{\eta}(\xi) \). is given by,

\[
\hat{u}(\xi, t) = e^{-i\xi ct} \hat{\eta}(\xi)
\]

Substituting for \( \hat{u}(\xi, t) \) in (**), we get

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi ct} \hat{\eta}(\xi) e^{i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi (x-ct)} dx = \eta(x-ct)
\]

This is equation for propagation of a wave solution.

(b) Repeat the same analysis for diffusion equation \( u_t = Du_{xx} \).
\[ u_t - Du_{xx} = 0 \quad \text{(Multiply by } e^{-i\xi x} \text{ and integrate from } -\infty \text{ to } \infty) \]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u_t - Du_{xx}) e^{-i\xi x} \, dx = 0
\]
\[
\hat{u}_t(\xi, t) + D\xi^2 \hat{u}(\xi, t) = 0
\]
\[
\hat{u}_t(\xi, t) = -D\xi^2 \hat{u}(\xi, t)
\]
which has a solution of the form for initial condition \( \hat{u}(\xi, 0) = \hat{\eta}(\xi) \)
\[
\hat{u}(\xi, t) = e^{-D\xi^2 t} \hat{\eta}(\xi)
\]
Substituting for \( \hat{u}(\xi, t) \) in (***) and using the result that Fourier transform of a gaussian function \[
\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}
\] is \( e^{-D\xi^2 t} \), we get
\[
u(x, t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}} \eta(x) \, dx
\]

**HINT**: Take the fourier transform of the respective equations

**Question 2: Semi-analytical solution**

Given the PDE for heat conduction,
\[ u_t = u_{xx} - \cos(2\pi x), \quad x \in \mathbb{R} \]

with boundary conditions \( u(t, 0) = u(t, 1) = 0 \) and Initial conditions \( u(0, x) = 0 \). Approximate the solution by applying the Method of Lines with the implicit Euler procedure for time integration. Formulating the method and then calculate an example with \( h = 0.01, t_f = 10 \).

\[
u_{xx} = \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2} + O(\Delta^2)
\]
\[x_i = ih; \quad u_i(t) = u(x_i, t)\]
\[
\hat{u}_t(t) = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2} - \cos(2\pi x_i; \quad i = 1, 2, \ldots, N - 1
\]
\[u_0(t) = u_N(t) = 0, \quad \text{and } u_l(0) = 0 \quad \text{for all } l
\]
\[\dot{u} = \frac{1}{h^2} \hat{A}u + b = Au + b \quad \text{(System of linear-equations)}\]

**Implicit-Euler:**
\[
\dot{u}^{n+1} = \dot{u}^n + \Delta t f(t + \Delta t, \dot{u}^{n+1})
\]
\[
\dot{u}^n + \Delta t (A\dot{u}^{n+1} + b) = \dot{u}^n + \frac{\Delta t}{\Delta x^2} \hat{A}\dot{u}^{n+1} + b\Delta t
\]
\[
\left( I - \frac{\Delta t}{\Delta x^2} \hat{A} \right) \dot{u}^{n+1} = \dot{u}^n + b\Delta t
\]
\[
\dot{u}^{n+1} = \left( I - \frac{\Delta t}{\Delta x^2} \hat{A} \right)^{-1} (\dot{u}^n + b\Delta t); \quad u^0 = 0
\]
where,

\[
A = \begin{pmatrix}
-2 & 1 & 0 & \ldots & \ldots & \ldots \\
1 & -2 & 1 & \ldots & \ldots & \ldots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ldots & \ddots & -2 & 1 & 0 & \ldots \\
\ldots & 1 & -2 & 1 & \ldots & \ldots \\
\ldots & \ldots & 0 & 1 & -2 & \\
\end{pmatrix} \in \mathbb{R}^{N-1 \times N-1}
\]

and \( b = [-\cos(2\pi h), -\cos(4\pi h), -\cos(6\pi h), \ldots, -\cos(2(N-1)\pi h)]^T \in \mathbb{R}^{N-1} \)

**Question 3: Programming Task**

Consider the poisson equation

\[
\Delta u(x, y) = 1 \quad (x, y) \in \Omega := (-1, 1) \times (-1, 1)
\]

\[u = 0 \quad \text{at } \partial \Omega\]

a) Discretize the problem with 5-point FD stencil.

b) Solve the linear system and plot the solution.