Question 1: Von Neumann-stability analysis

\[ u_t = \alpha u_{xx} \]

a) Show, using the Von Neumann-stability analysis, that the Crank-Nicolson method applied to the heat equation with central finite differences in space, is unconditionally stable

Solution:
The numerical discretization using Crank-Nicolson method for time-integration and central-difference for space derivatives looks like

\[ u_i^{n+1} = \Delta t u_i^n + \frac{\alpha \Delta t}{2 \Delta x^2} \left( u_{i-1}^n - 2u_i^n + u_{i+1}^n \right) + \frac{\alpha \Delta t}{2 \Delta x^2} \left( u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1} \right) \]

For linear-problems, we can show that the error at position \( i \) and time \( n \), evolves according to

\[ e_i^{n+1} = \Delta t e_i^n + \frac{\alpha \Delta t}{2 \Delta x^2} \left( e_{i-1}^n - 2e_i^n + e_{i+1}^n \right) + \frac{\alpha \Delta t}{2 \Delta x^2} \left( e_{i-1}^{n+1} - 2e_i^{n+1} + e_{i+1}^{n+1} \right) \]

Expanding error as Fourier series \( e(x) = \sum_{m=1}^{M} e^{at} e^{ik_m x} \), substituting for the above form for single \( k \) yields

\[ e^{at} = 1 + \frac{\alpha \Delta t}{2 \Delta x^2} \left( e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right) + \frac{\alpha \Delta t}{2 \Delta x^2} \left( e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right) \]

Using Euler-formula \( e^{ix} = \cos(x) + i \sin(x) \), we can show that

\[ \sin\left( \frac{k_m \Delta x}{2} \right) = \frac{e^{ik_m \Delta x/2} - e^{-ik_m \Delta x/2}}{2i} \implies \sin^2\left( \frac{k_m \Delta x}{2} \right) = -\frac{e^{ik_m \Delta x/2} + e^{-ik_m \Delta x/2} - 2}{4} \]

Substituting in (*),

\[ |e^{at}| = \left| \frac{e^{n+1}}{e^n} \right| = \left| 1 + \frac{2\alpha \Delta t}{\Delta x^2} \sin^2\left( \frac{k_m \Delta x}{2} \right) \right| \leq 1 \]

So, the amplification factor is always bounded by 1. So, Crank-Nicolson with central difference for space-derivatives is unconditionally stable.
b) In a similar way, show that the Leap-frog method applied to above equation is unconditionally unstable.

**Solution:**
The discretized numerical solution with leap-frog time-stepping looks like

\[ u_{i}^{n+1} = u_{i}^{n-1} + 2r \left( u_{i-1}^{n} - 2u_{i}^{n} + u_{i+1}^{n} \right) \]

where \( r = \frac{\Delta t}{\Delta x^2} \).
The corresponding error evolution equation is

\[ e_{i}^{n+1} = e_{i}^{n-1} + 2r \left( e_{i-1}^{n} - 2e_{i}^{n} + e_{i+1}^{n} \right) \]

Repeating the same analysis as before, we get the following quadratic \( g = e^{at} \),

\[ g^2 + 8rg \sin^2 \left( \frac{k \Delta x}{2} \right) - 1 = 0 \]

The roots of the equation are,

\[ g_{\pm} = -4r\sin^2 \left( \frac{k \Delta x}{2} \right) \pm \sqrt{16r^2\sin^4 \left( \frac{k \Delta x}{2} \right) + 1} \]

Consider the cases,

- case 1: \( r = 1 \) and \( k \Delta x = \pi, |g_{-1}| > 1 \),
- case 2: \( r > 1 \) and \( k \Delta x = \pi, |g_{-1}| > 1 \)
- case 3: \( r < 1 \), stable for some \( r \)

**Question 2: Hyperbolic equation**
Consider the PDE for advection equation

\[ u_t + cu_x = 0 \]

Show that for the CTCS-method (Leapfrog?) the local truncation error is of the form

\[ \text{error} = -\frac{1}{6} \Delta t^2 u_{ttt} |i|^n - \frac{c}{6} \Delta x^2 u_{xxx} |i|^n + \text{H.O.T in } \Delta t \text{ and } \Delta x \]

**Solution:**
The CTCS discretization for the advection equation,

\[ \frac{u_{i}^{n+1} - u_{i}^{n-1}}{2\Delta t} + c \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x} = 0 \ (\ast) \]

Taylor expanding around the solution \((x_i, t^n)\), we obtain,

\[ u_{i}^{n+1} = u_{i}^{n} \pm \Delta t u_{i}^{n} + \frac{\Delta t^2}{2} u_{tt} |i|^n \pm \frac{\Delta t^3}{3!} u_{ttt} |i|^n + \text{H.O.T in } \Delta t \]

\[ u_{i}^{n} \pm \Delta x u_{x} |i|^n + \frac{\Delta x^2}{2} u_{xx} |i|^n \pm \frac{\Delta x^3}{3!} u_{xxx} |i|^n + \text{H.O.T in } \Delta x \]
Substituting the above expansions into the difference equation (*), we obtain.

\[
\frac{2\Delta t u^n_i}{\Delta t} + \frac{\Delta t^3}{3!} u^{n+1}_i + \text{H.O.T in } \Delta t + c \frac{2\Delta x u^n_j}{\Delta x} + \frac{\Delta t^3}{3!} u^{n+1}_j + \text{H.O.T in } \Delta x
\]

The PDE satisfies the \((u_t + cu_x)|_i^n = 0\), simplifying the above expression to

\[
\tau = -\frac{1}{6}\Delta t^2 u^{n+1}_i - \frac{c}{6}\Delta x^2 u^{n+1}_j + \text{H.O.T in } \Delta t \text{ and } \Delta x
\]

**Question 3: Stability of hyperbolic PDEs**

Work out the Von Neumann stability analysis for the wave equation with the CTCS scheme

\[
u_{tt} = c^2 u_{xx}
\]

\[
\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (CTCS)
\]

**Solution:**

the error propagation equation looks like,

\[
e_j^{n+1} - 2e_j^n + e_j^{n-1} = p^2(e_{j+1}^n - 2e_j^n + e_{j-1}^n) \quad (CTCS)
\]

where \(p^2 = \frac{c^2\Delta t^2}{\Delta x^2}\). Repeating similar analysis as in previous problems,

\[
g^2 - 2g + 1 = -4p^2 g \sin^2 \left(\frac{k\Delta x}{2}\right)
\]

\[
g^2 - 2g(1 - 2p^2) \sin^2 \left(\frac{k\Delta x}{2}\right) + 1 = 0
\]

The roots of the equation are,

\[
g_\pm = 1 - 2p^2 \sin^2 \left(\frac{k\Delta x}{2}\right) \pm \sqrt{4p^2 \sin^2 \left(\frac{k\Delta x}{2}\right) \left[p^2 \sin^2 \left(\frac{k\Delta x}{2}\right) - 1\right]}
\]

Let us consider three cases,

- **case 1:** \(p^2 < 1\): \(|g_\pm| < 1\),
- **case 2:** \(p^2 = 1\): \(|g_\pm| = 1\), Hence the scheme is stable for \(p^2 \leq 1\)
- **case 3:** \(p^2 > 1\), consider the scenario, \(k\Delta x = \pi\),

\[
g_\pm = 1 - 2p^2 \pm 2p \sqrt{p^2 - 1}
\]

So \(q_{-1} < 1 - 2p^2 < -1\), \(\forall p^2 > 1\), thus \(|g_{-1}| > 1\) at \(k\Delta x = \pi\)

So the CTCS scheme is only stable for \(p^2 < 1\)
**Question 4: Mass conservation**

Show that for the non-linear hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial [F(u)]}{\partial x} = 0 \quad (*)$$

the following property holds

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx \quad \forall t \geq 0$$

if we assume that \( \lim_{x \to \pm \infty} F(u(x,t)) = 0, \forall t \geq 0 \)

**Solution**

Integrating the equation (*) over the interval \([-\infty, \infty] \times [0, t]\) and using the fundamental theorem of calculus, we obtain

$$\int_{0}^{t} \int_{-\infty}^{\infty} (u_t + [F(u)]_x) \, dx \, dt = \int_{-\infty}^{\infty} [u(x,t)-u(x,0)]dx + \int_{0}^{t} [F(u(\infty,t))-F(u(-\infty,t))]dt = 0$$

Taking the fluxes to be zero at infinity i.e. \( \lim_{x \to \pm \infty} F(u(x,t)) = 0, \forall t \geq 0 \), we have

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx, \forall t \geq 0$$